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# APPLICATION OF BOUNDARY BEHAVIOUR TO THE FUNCTIONS OF BOUNDED BOUNDARY ROTATION

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## 1. Introduction

We say $f \in V_k$ if $f(z)$ is regular in the unit disk $\Delta = \{z : |z| < 1\}$, $f'(z) \neq 0$ in $\Delta$, $f$ is normalized so that $f(0) = 0, f'(0) = 1$, and if for some real number $k \geq 2$,

$$\sup_{k<1} \int_0^{2\pi} |Re(1 + zf''(z)/f'(z))| d\theta \leq k\pi, \; (z = re^{i\theta}). \tag{1.1}$$

That is, we call $V_k$ the set of functions $f$ which are analytic and locally univalent in $\Delta$ that have boundary rotation no greater than $k\pi$. (1.1) is equivalent to having

$$f''(z)/f'(z) = \int_0^{2\pi} (e^{it} - z)^{-1} d\mu(t) \tag{1.2}$$

where $\mu(t)$ is a function of bounded variation such that $\int_0^{2\pi} |d\mu(t)| \leq k\pi$ and $\int_0^{2\pi} d\mu(t) = 2\pi$. Functions in $V_k$ form a linear invariant family of order $\frac{k}{2}$.

In this paper we introduce the notion of "even well accessibility in an arc" and we apply some of results obtained in [6] and [7] to functions of bounded boundary rotation $V_k (k \geq 2)$.

## 2. Preliminary Remark

We begin with the following lemma about boundary family of functions in $V_k$. We refer the definition of boundary family $C(e^{i\theta}, f)$ in [6, p.84].

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LEMMA. *If $f$ belongs to $V_k$, then $C(e^{i\theta}, f)$ consists of the single Koebe function $g_c(z)$ with $c = c(\theta) \equiv \mu(\theta^+) - \mu(\theta^-) - 1$.*

*Proof.* A standard calculation with (1.2) shows that

$$\lim_{z\to 1}(1-z)e^{i\theta}f''(ze^{i\theta})/f'(ze^{i\theta}) = \mu(\theta^+) - \mu(\theta^-)$$

for $z$ in a Stolz angle. The conclusion follows from Theorem 3. 2[6, p.91]

REMARK. This lemma immediately yields some results about the geometry of functions in $V_k$. First, for all $\theta \in [0, 2\pi)$ we have

$$\log f'(ze^{i\theta})/\log(1-z)^{-1} \to c(\theta) + 1 \tag{2.1}$$

as $z \to 1$ in a Stolz angle. Second, if $\omega \to 1$ in an angle so that $\frac{1-|\omega|}{1-|z|}$ remains bounded above and below, then for all $\theta$ one has

$$\left(\frac{1-\omega}{1-z}\right)^{c(\theta)+1} \frac{f'(\omega e^{i\theta})}{f'(ze^{i\theta})} \to 1. \tag{2.2}$$

Both of these results follow from our Lemma and Folgerung 3.10 in [9].

If we suppose that $f \in V_k$ then $f(ze^{i\theta})$ approaches a finite limit $f(e^{i\theta})$ as $z \to 1$ in an angle for all $\theta \in [0, 2\pi)$ with the exception of at most $[\frac{k}{2}+1]$ values where [4] denotes the greatest integer function. Furthermore if $f \in V_k$ then for all but at most $[\frac{k}{2}+1]$ exceptional values of $\theta$, we have

$$\frac{f(e^{i\theta}) - f(ze^{i\theta})}{(1-z)f'(ze^{i\theta})} \to -\frac{1}{c(\theta)}, \tag{2.3}$$

and

$$1 \geq \frac{d(f(x))}{|f(e^{i\theta}) - f(xe^{i\theta})|} \to \begin{cases} 1, & -\infty < c(\theta) \leq -1 \\ \sin\frac{\pi|c|}{2}, & -1 < c(\theta) < 0 \end{cases} \tag{2.4}$$

where $d(f(x))$ denotes the radius of the largest schlicht disk centered at $f(x)$ that lies on the Riemann surface $f(\Delta)$. In (2.3) $z \to 1$ in a Stolz angle and in (2.4) $x$ approach 1 along the real axis. The first result on angular limits of $V_k$ functions is well known [4].

The preceding three results will follow from Satz 3.4, 3.14 and Folgerung 3.11 of [9] as soon as it is shown that $c(\theta) \geq 0$ occurs for no more than $[\frac{k}{2}+1]$ values of $\theta$. Places where $c(\theta) \geq 0$ correspond to points where $\mu(t)$ in (1.2) has positive jumps equal to or greater than one. There are at most $[\frac{k}{2}+1]$ such points since the total positive variation of $\mu(t)$ cannot exceed $\frac{k}{2}+1$. In a similar vein the limits (2.3) and (2.4) are both equal to one for all but at most a countable set of points.

## 3. Main Results

If $f$ belongs to $V_k$ then with a detailed analysis involving Lebesgue points of the representing function $\mu(t)$ in (1.2) one can show that $f$ is conformal on the boundary of $\Delta$ except perhaps on a countable set of points. It seems natural then to expect that the set of semiconformal points on $\partial\Delta$ must be even larger, but this is false. Indeed with (2.3) one can show that $f \in V_k$ may fail to be semiconformal on a countably infinite subset of $\partial\Delta$.

Paatero [5] showed for univalent functions in $V_k$ that

$$\frac{f(e^{i\theta}) - f(ze^{i\theta})}{1-z} \tag{3.1}$$

tends to a finite limit as $z \to 1$ in an angle for almost all $\theta$. Relation (2.3) holds for all but at most $[\frac{k}{2}+1]$ points and displays the limit explicitly. It is not possible to rewrite (2.3) in a way that extends the existence of angular limits of (3.1) to all but a countable set because it is possible for $f'(z)$ to have infinite radial limits on an uncountable set when $f$ belongs to $V_k$. Indeed, if $f \in V_k$ then a standard representation in terms of starlike functions $S_1, S_2$, is [1]

$$f'(z) = (S_1(z)/z)^{(k+2)/4}(S_2(z)/z)^{(k-2)/4}.$$

Seidel [10] has constructed explicitly an example of a starlike function with infinite radial limits on an uncountable set and this provides the example in question. Of course (2.3) is true for the entire family $V_k$ without the restriction of univalence imposed by Paatero.

The notion of uniform well accessibility appears in [2]. This notion has the disadvantage of involving unrestricted approach to the boundary of $\Delta$. With this in mind we introduce a related notion for which there exist simple, easily verifiable, sufficient analytic conditions. First, if $E$ is an arc of the unit circle, $|z| = 1$, we let $C(E, f)$ denote the union of $C(e^{i\theta}, f)$ for all $\theta$ in $E$.

DEFINITION. *A function $f(z)$ in $U_\alpha$ is said to be evenly well accessible on an arc, $E$, if there are number $\delta > 0, B < \infty$ such that for all $g \in C(E, f)$*

$$(1-x^2)|g'(x)| \le B(1-x)^\delta,\ 1 \le x < 1. \tag{3.2}$$

*We shall see that "even well accessibility on arcs" provides all of the interesting conclusions that follows from the concept of "uniform well accessibility" and has the obvious advantage of being computationally verifiable. Unlike $C(e^{i\theta}, f)$ or $C(f)$ [9] the set $C(E, f)$ can be disconnected and can have any finite cardinality or be countably infinite.*

THEOREM 3.1. *If $f \in U_\alpha$ is evenly well accessible on an arc $E \subset \partial\Delta$, then $f$ is Hölder continous on $E$, that is*

$$|f(z) - f(\omega)| \le B|z - \omega|^\delta, \quad z, \omega \in E. \tag{3.3}$$

*Proof.* A careful examination of the proof for Satz 3.4 in [9] shows that with the uniformity given in (3.2) one can prove

$$|f(z)| \le B_1(1 - |z|)^{\delta - 1}$$

for all $z$ in the sector of $\Delta$ that $E$ subtends at 0. The Hölder continuity on $E$ then follows by a standard argument [2].

THEOREM 3.2. *In order that $f \in U_\alpha$ be evenly well accessible on an arc $E \subset \partial\Delta$ it is sufficient that $C(e^{i\theta}, f)$ have Case I [6, p.86] behaviour at a finite number of points $e^{i\theta} \in E$ and that for the remaining points of $E, C(e^{i\theta}, f) = g_c$ where $c$ may vary over $E$ but $\sup\{Re\, c\} < 0$.*

*Proof.* Since $|g_c'(x)| = (1 + x)^{a-1}/(1 - x)^{a+1}$ where $a = Re(c)$, it is clean that

$$(1 - x^2)|g_c'(x)| \le (1 - x)^{\delta_1}, \; 0 \le x < 1. \tag{3.4}$$

(where $\delta_1 = -\sup Re(c) > 0$). By the definition of Case I behaviour one has the following for the finite number of exceptional points $e^{i\theta} \in E$:

$$(1 - x^2)|g'(x)| \le B(\theta)(1 - x)^{\delta(\theta)}, \; 0 \le x < 1. \tag{3.5}$$

where $B(\theta)$ and $\delta(\theta)$ depend only on $\theta$. By hypothesis the number of exceptional points of $E$ is finite. It then follows from the definition of $C(E, f)$ as the union of $C(e^{i\theta}, f)$ that one can combine (3.4) and (3.5) to find a single pair of constants $B$ and $\delta$ which fulfill the definition.

COROLLARY. *If $f$ belong to $V_k$ then $f$ is continuous on $\partial\Delta$ with the exception of at most $[\frac{k}{2} + 1]$ points. Furthermore $f$ is Hölder continuous on any closed subarc of $\partial\Delta$ that contains no exceptional points.*

*Proof.* This result follows immediately from Lemma and Theorems 3.1 and 3.2.

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