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LARGE DEVIATION PRINCIPLES OF RANDOM VARIABLES OBTAINED BY SCALING RANDOM MEASURE

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1. Introduction

Let $\{X_n : n \geq 1\}$ be an ergodic sequence of random variables with the same mean $E[X_n] = m$ for each integer n. Let $S_n = X_1 + \cdots + X_n$ be the partial sums. If $E[X_n]$ is finite for each n, S_n/n converges to m by the law of large numbers. We set $M_n(t) = E[\exp(tS_n)]$, the moment generating function, which will be assumed to exist for every real number t. We say that S_n/n satisfies the large deviation principle with rate function $I(x)$, where $I(x)$ is non-negative, lower semi-continuous and convex function, if

for any closed subset F in R,

$$\limsup_{n\to\infty} \frac{1}{n} \cdot \log P\left[\frac{S_n}{n} \in F\right] \leq -I(F), \tag{1.1}$$

and,

for any open subset G in R,

$$\liminf_{n\to\infty} \frac{1}{n} \cdot \log P\left[\frac{S_n}{n} \in G\right] \geq -I(G). \tag{1.2}$$

Cramér presented the first large deviation theorem at a probability symposium in 1937. Since 1937, this theory has undergone an extensive development and this original work was extended in various directions. There have been many developments in the theory of large deviations over the last two decades.

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Primary and most significant is the work of M. D. Donsker and S. R. S. Varadhan [1, 2] who have developed a powerful machinery in a series of papers to deal with many old and new problems in probability where precise estimates of the large deviation probabilities play an important role. Gärtner [5] and Ellis [4] have developed useful and surprising generalizations with assumptions about the dependence of the random variables and the moment generating functions. Also the large deviation principles have found many applications in statistics (Groeneboom [6]), statistical mechanics [Ellis [3], Lanford [8]) and in stochastic processes [1, 2].

Throughout this paper, we study the large deviation principles for distributions of scaling limits of random measures. That is to say, we give computation of the large deviations. Also, this property is possessed by the following classes of examples: Poisson point process, Poisson center cluster random measure and doubly stochastic process.

2. Preliminaries and Main Theorem

We assume that the readers are familiar with the language of random measures. As a good reference for details, Kallenberg's book [7] may be consulted. N will denote the set of Radon Borel measures on R^d, so that if $\mu \in N$ then it is finite on bounded Borel sets. Let η be the σ-algebra of subsets of N generated by sets of the form $\{\mu \in N \mid \mu(B) < r\}$ for a bounded Borel set B and a non-negative real number r. A random measure is a measurable function $X : (\Omega, \Sigma, P) \to (N, \eta)$, where (Ω, Σ, P) is a fixed probability space. If B is a Borel set, then we let $\mu(B)$ be the random amount of mass the measure μ gives to B.

From now on, let X be a random measure on R^d. Let A be a bounded Borel subset of R^d. All random measures will be assumed to be stationary, i.e., with a translation invariant distribution. The most well known random measure is the Poisson point process with intensity $\alpha > 0$ (see Definition 3).

Let A be a Borel subset of R^d. We denote $X_r(A)$ by $X_r(A) = X(rA)$ for $r \in R^+$. The ergodic theorem says that $X_r(A)/r^d$ converges to a mean of $X_r(A)/r^d$ as $r \to \infty$. We will show that the random variable obtained by scaling a random measure X, $X_r(A)/r^d$ satisfies the large deviation principle with rate function $I(x)$. That is, (large deviation upper bound) ; For any closed subset F in R,

$$\limsup_{r\to\infty}\frac{1}{r^d}\cdot\log\left[\frac{X_r(A)}{r^d}\in F\right]\le -I(F). \tag{2.1}$$

(large deviation lower bound) ; For any open subset G in R,

$$\liminf_{r\to\infty}\frac{1}{r^d}\cdot\log P\left[\frac{X_r(A)}{r^d}\in G\right]\ge -I(G). \tag{2.2}$$

In the above, we denote, for each Borel subset A in R^d,

$$I(B)=I_A(B)=\inf_{x\in B}I_A(x)=\inf_{x\in B}I(x) \text{ for a Borel subset } B \text{ in } R,$$

where

$$I(x)=I_A(x)=\sup_{t\in R}\{tx-M(t)\},$$

and

$$M(t)=\lim_{r\to\infty}\frac{1}{r^d}\cdot\log E[e^{tX_r(A)}].$$

Let us consider a random variable $X_r(A)$ for a bounded Borel subset A of R^d. The moment generating function $M_r(t)$ of a random variable $X_r(A)$ is defined by

$$M_r(t)=(M_A)_r(t)=E[e^{tX_r(A)}] \text{ for every } t\in R.$$

Also,

$$m_r(A)=E[X_r(A)] \text{ for a bounded Borel subset } A \text{ of } R^d.$$

Assume that

(2.3) (a) $C_r(t)=\frac{1}{r^d}\cdot\log M_r(t)$ is finite for every $t\in R$,

(2.4) (b) $M(t)=\lim_{r\to\infty}C_r(t)$ exists and is finite for every $t\in R$,

and

(2.5) (c) $m(A)=\lim_{r\to\infty}\frac{1}{r^d}\cdot m_r(A)$ exists for a bounded Borel set A of R^d.

Define the Legendre-Fenchel transformation of the convex function $M(t)$ by

$$I(x)=\sup_{t\in R}\{tx-M(t)\}. \tag{2.6}$$

Now, the main results in this paper are as follows;

THEOREM 1. *Under assumptions (2.3), (2.4) and (2.5),*

(i) *$I(x)$ is convex, non-negative, and lower semi-continuous.*
(ii) *I has compact level sets.*
(iii) *The large deviation upper bound, i.e., the formula (2.1) is valid.*
(iv) *If $M(t)$ is differentiable for every $t \in R$, then the large deviation lower bound, i.e., the formula (2.2), is valid.*

Proof. (i) For every x_1 and x_2 in R and $\lambda \in [0,1]$,

$$\begin{aligned} I\{\lambda x_1 + (1-\lambda)x_2\} &= \sup_{t\in R}\{\lambda t x_1 + (1-\lambda)t x_2 - M(t)\} \\ &= \sup_{t\in R}\{\lambda[t x_1 - M(t)] + (1-\lambda)[t x_2 - M(t)]\} \\ &\le \lambda \cdot \sup_{t\in R}\{t x_1 - M(t)\} + (1-\lambda)\cdot \sup_{t\in R}\{t x_2 - M(t)\} \\ &= \lambda \cdot I(x_1) + (1-\lambda)\cdot I(x_2). \end{aligned}$$

Thus, $I(x)$ is convex.

If $t=0$, then $tx - M(t) = 0$ for every $x \in R$. Since $I(x) \ge 0$, $I(x)$ is non-negative.

Since $tx - M(t)$ is continuous for x, $I(x)$ is lower semi-continuous.

(ii) For every real number $k \ge 0$, let $K_k = \{x \in R : I(x) \le k\}$. $(K_k)^c$ is open since $I(x)$ is lower semi-continuous, i.e., $\{x \in R : I(x) > k\}$ is open. Thus K_k is closed. On the other hand, let's take $x \in K_k$. Since $I(x) \le k$, $\{x : I(x) \le k\} \subseteq [-k-M(-1),\ k+M(1)]$. K_k is also bounded. Thus the set $\{x \in R : I(x) \le k\}$ is compact.

(iii) Since $tx - M(t) = 0$ for $t = 0$, so $I(x) \ge 0$. If $E[X_r(A)] = m_r(A)$, and $m(A) = \lim_{r\to\infty} \frac{1}{r^d}\cdot m_r(A)$ exists (denote $m = m(A)$), then Jensen's inequality implies that $E[e^{tX_r(A)}] \ge e^{E[tX_r(A)]} = e^{tm_r(A)}$ for all t. So, $tm - M(t) \le 0$ for all t and $I(m) \le 0$. Therefore, if the limit m of $m_r(A)$, exists, then $I(m) = 0$.

By the property (i), since $I(x)$ is convex, $I(x)$ is non-increasing on $(-\infty, m]$ and non-decreasing on $[m, \infty)$.

Note that both $x < m$ and $t > 0$ imply $xt - M(t) < mt - M(t) \le 0$, and both $x > m$ and $t < 0$ imply $xt - M(t) < mt - M(t) \le 0$.

Therefore, $I(x)$ need be taken only over both $t > 0$ for $x > m$ and $t < 0$ for $x < m$.

For $t > 0$; let $I_1 = [a, \infty)$ for $a > m$.

$$Q_r(I_1) = P\left[\frac{X_r(A)}{r^d} \le a\right] \le e^{-ar^d t} \cdot E\left[e^{tX_r(A)}\right]$$

by Chebychev's inequality

$$= e^{-r^d[at - C_r(t)]}.$$

So,

$$\limsup_{r\to\infty} \frac{1}{r^d} \cdot \log Q_r(I_1) \le -\sup_{t>0}\{at - M(t)\}.$$

For $t < 0$; let $I_2 = (\infty, a]$ for $a < m$.

$$Q_r(I_2) = P\left[\frac{X_r(A)}{r^d} \le a\right] \le e^{-ar^d t} \cdot E\left[e^{tX_r(A)}\right]$$

by Chebychev's inequality

$$= e^{-r^d[at - C_r(t)]}.$$

So,

$$\limsup_{r\to\infty} \frac{1}{r^d} \cdot \log Q_r(I_2) \le -\sup_{t<0}\{at - M(t)\}.$$

Now, in order to prove the main property in general, let F be an arbitrary closed set.

If $m \in F$, then $I(F) = 0$ and so

$$\limsup_{r\to\infty} \frac{1}{r^d} \cdot \log Q_r(F) \le 0.$$

If $m \notin F$, let (a_1, a_2) be the largest interval containing m such that

$$F \cap (a_1, a_2) = \phi.$$

Note that $I(x)$ is non-increasing on $(-\infty, a_1]$ and non-decreasing on $[a_2, \infty)$.

So,

$$\inf_{a\in F} I(a) = \min[I(a_1), I(a_2)].$$

Case(i); $F \subseteq [a_2, \infty)$ and so the property (iii) is trivial.
Case(ii); $F \subseteq (-\infty, a_1]$ and so the property (iii) is trivial.
Case(iii); $F \subseteq (-\infty, a_1] \cup [a_2, \infty)$.

$$\begin{aligned} Q_r(F) &\le Q_r\{(-\infty, a_1] \cup [a_2, \infty)\} \\ &= Q_r\{(-\infty, a_1]\} + Q_r\{[a_2, \infty)\} \\ &\le \exp\{-r^d \cdot I(I_1)\} + \exp\{-r^d I(I_2)\} \\ &\le 2\exp\{-r^d \cdot \min[I(I_1), I(I_2)]\}. \end{aligned}$$

Therefore, the result holds in this case also. Throughout Case (i), (ii) and (iii), the large deviation upper bound holds.

(iv) Now, it remains to prove the property (iv).

Let $Q_r(dx)$ denote the distribution of $X_r(A)/r^d$ on R. We define probability measures, for any real number t,

$$Q_{r,t}(dx) = \frac{\exp(r^d tx)}{\exp\{-r^d C_r(t)\}} \cdot Q_r(dx) \text{ for positive real number } r.$$

Let G be an open set in R and x_0 any point in G. We take a neighborhood $G_e(x_0) = (x_0 - e, x_0 + e)$ such that $G_e(x_0) \subset G$ for $e > 0$. If x ranges over $G_e(x_0)$, then $-tx \ge -tx_0 - |t|e$. Since $G_e(x_0) \subset G$, we have

$$\begin{aligned} Q_r(G) &\ge Q_r(G_e(x_0)) \\ &= \int_{G_e(x_0)} Q_r(dx) \\ &= \exp\{r^d C_r(t)\} \int_{G_e(x_0)} \exp(-r^d tx) \cdot Q_{r,t}(dx) \\ &\ge \exp\{r^d [C_r(t) - tx_0 - |t|e]\} \cdot Q_{r,t}\{G_e(x_0)\}. \end{aligned}$$

Thus,

$$\begin{aligned} &\liminf_{r\to\infty} \frac{1}{r^d} \cdot \log Q_r(G) \\ &\qquad \ge C(t) - tx_0 - |t|e + \liminf_{r\to\infty} \frac{1}{r^d} \cdot \log Q_{r,t}\{G_e(x_0)\} \\ &\qquad = C(t) - tx_0 - |t|e \quad \text{by Lemma 2} \\ &\qquad \ge -I(x_0) - |t|e \quad \text{by the definition of } \; I(x). \end{aligned}$$

If we take $e \to 0$ since e is arbitrary, then we have

$$\liminf_{r\to\infty} \frac{1}{r^d} \cdot \log Q_r(G) \geq -I(x_0).$$

Since we take an arbitrary point x_0 in G, we can take $I(G) = \inf_{x_0 \in G} I(x_0)$ instead of $I(x_0)$.

Finally, we have that

$$\liminf_{r\to\infty} \frac{1}{r^d} \cdot \log Q_r(G) \geq -I(G) \text{ for any open set } G \text{ in } R.$$

Thus, we complete proof of the property (iv).

To complete our proof, we need the following lemma 2.

LEMMA 2. *Assume hypotheses (2.3) and (2.4). $C(t)$ is differentiable at a point t_0 with $C'(t_0) = x_0$. For any $e > 0$,*

$$\lim_{r\to\infty} Q_{r,t(0)}\{G_e(x_0)\} = 1.$$

Proof. Now, we introduce a sequence $\{W_{r,t(0)}; r \text{ is real}\}$ of random variables such that $W_{r,t(0)}/r^d$ has distribution $Q_{r,t(0)}(dx)$ (We denote $t_0 = t(0)$).

$$\begin{aligned}
&P\left[\frac{W_{r,t(0)}}{r^d} \geq x_0 + e\right] = P\left[\frac{W_{r,t(0)}}{r^d} - x_0 - e \geq 0\right] \\
&\leq E\left[\exp\{r^d t(\frac{W_{r,t(0)}}{r^d} - x_0 - e)\}\right] \text{ by Markov' inequality for } t \geq 0 \\
&\leq \exp\{-r^d t(x_0 + e)\} \cdot E\left[\exp\{r^d t(\frac{W_{r,t(0)}}{r^d})\}\right] \\
&= \exp\{-r^d t(x_0 + e)\} \int \exp(r^d t x) \cdot Q_{r,t(0)}(dx) \\
&= \exp\{-r^d t[x_0 + e - r^d C_r(t_0)]\} \int \exp\{r^d(t + t_0)x\} \cdot Q_r(dx) \\
&= \exp\{-r^d t[x_0 + e - r^d C_r(t_0)]\} \cdot \exp\{r^d C_r(t + t_0)\}
\end{aligned}$$

since $M_r(t) = \int \exp(r^d tx) \cdot Q_r(dx) = \exp\{r^d C_r(t)\}$ by the definition of $C_r(t)$. That is,

$$P\left[\frac{W_{r,t(0)}}{r^d} \geq x_0 + e\right] \leq \exp\{r^d[C_r(t_0+t) - C_r(t_0) - t(x_0+e)]\}.$$

Thus,

$$\limsup_{r\to\infty} \frac{1}{r^d} \cdot \log P\left[\frac{W_{r,t(0)}}{r^d} \geq x_0 + e\right] \leq C(t_0+t) - C(t_0) - t(x_0+e).$$

Similarly,

$$\limsup_{r\to\infty} \frac{1}{r^d} \cdot \log P\left[\frac{W_{r,t(0)}}{r^d} \leq x_0 - e\right] \leq C(t_0-t) - C(t_0) + t(x_0-e).$$

The both right sides are strictly negative for small t since $C(t)$ is differentiable, i.e., $C'(t_0) = x_0$. Both imply that $Q_{r,t(0)}\{G_e(x_0)\} \to 1$ as $r \to \infty$.

3. Applications

3.1. Poisson random measure with intensity $\alpha > 0$

Let X be a Poisson point process with intensity $\alpha > 0$. Let A be a bounded Borel subset of R^d. The moment generating function of the random variable $X_r(A)$ is

$$M_r(t) = E\left[e^{tX_r(A)}\right] = e^{\alpha r^d |A|(e^t - 1)} \text{ for every } t \in R, \tag{3.1}$$

where $|A|$ means Lebesgue measure of A.

Also we have

$$M(t) = \lim_{r\to\infty} \frac{1}{r^d} \cdot \log M_r(t) \text{ and } I(x) = \sup_{t\in R}\{tx - M(t)\}. \tag{3.2}$$

Now, for a Poisson point process X, we will show that $X_r(A)/r^d$ satisfies the large deviation principle with rate function $I(x)$.

DEFINITION 3. X is called a Poisson point process with intensity $\alpha > 0$ if

(1) $X(A)$ is a Poisson random variable with parameter $\alpha|A|$ for every bounded Borel subset A in R^d and
(2) If $A_1, A_2, \ldots, A_n$ are disjoint bounded Borel subsets of R^d, then $X(A_1), X(A_2), \ldots, X(A_n)$ are independent Poisson random variables with respective parameters $\alpha|A_1|, \alpha|A_2|, \ldots, \alpha|A_n|$, where $|\cdot|$ denotes Lebesgue measure.

THEOREM 4. *Let X be a Poisson random measure with intensity $\alpha > 0$. Then*

(a) $M(t) = \alpha|A|(e^t - 1)$ *and*

$$I(x) = \sup_{t \in R}\{tx - M(t)\} = x \cdot \log\left[\frac{x}{\alpha|A|}\right] - x + \alpha|A|.$$

(b) $X_r(A)/r^d$ *satisfies the large deviation principle with rate function*

$$I(x) = x \cdot \log\left[\frac{x}{\alpha|A|}\right] - x + \alpha|A|.$$

Proof. (a) Since $M_r(t) = \exp\{\alpha r^d|A|(e^t - 1)\}$,

$$\begin{aligned} M(t) &= \lim_{r \to \infty} \frac{1}{r^d} \cdot \log M_r(t) \\ &= \alpha|A|(e^t - 1). \end{aligned}$$

So, $I(x) = \sup_{t \in R}\{tx - M(t)\} = \sup_{t \in R}\{tx - \alpha|A|(e^t - 1)\}$.

Now, to find $I(x)$, let $f(t) = tx - \alpha|A|(e^t - 1)$. Differentiating $f(t)$ with respect to t and then solving in terms of x,

$$\frac{df}{dt} = x - \alpha|A|e^t = 0. \quad \text{So} \quad \frac{x}{\alpha|A|} = e^t.$$

Note that each x is positive. Thus we get that $t = \log\left[\frac{x}{\alpha|A|}\right]$.

Thus,

$$I(x) = x \cdot \log\left[\frac{x}{\alpha|A|}\right] - x + \alpha|A|.$$

(b) Moreover, obviously $M(t)$ is differentiable for every real number t and thus $X_r(A)/r^d$ satisfies the large deviation principle with rate function $I(x)$ given as above by Theorem 1.

3.2. Poisson center cluster random measure

Let U be a stationary Poisson process on R^d with intensity $\alpha > 0$. V is a point process which has finite total expected mass, $\zeta = E[V(R^d)]$. Let x_i be the random occurrences of U and let $\{V_i\}$ be independent identically distributed (i.i.d.) copies of V that are also independent of U.

DEFINITION 5. The resulting cluster process X is said to be a Poisson center cluster process, which is defined by superimposing i.i.d. copies of V centered at the occurrences of U.

In other words. If A is a bounded Borel subset of R^d, then X is defined by

$$X(A) = \sum_{x_i} V_i(A - x_i). \tag{3.3}$$

The moment generating function of $V(R^d)$ is

$$M(t) = E\left[e^{tV(R^d)}\right] \quad \text{for every real number } t. \tag{3.4}$$

In addition, we assume that the moment generating function $M(t)$ is finite for every real number t.

Note that $E[X(A)] = \alpha\zeta|A|$.

In the case of a Poisson center cluster random measure X, let us consider the random variable obtained by scaling such random measure X, that is, $X_r(A)/r^d$ for a bounded Borel subset A of R^d. The ergodic theorem says that $X_r(A)/r^d$ converges to $\alpha\zeta|A|$ as $r \to \infty$. Then we have the following property.

THEOREM 6. *X is a Poisson center cluster random measure. Let A be a bounded Borel subset of* R^d. *Then* $X_r(A)/r^d$ *satisfies the large deviation principle with rate function* $I(x) = \sup_{t\in R}\{tx + \alpha|A| - \alpha|A|M_{V(R^d)}(t)\}$.

Proof. First, let us consider the moment generating function of $X_r(A)$

$$\begin{aligned}
M_r(t) &= E\left[e^{tX_r(A)}\right] \\
&= E\left[e^{tX(rA)}\right] \\
&= E\left[e^{t\Sigma_i V_i(rA-x_i)}\right] \\
&= E\left[E\left[e^{t\Sigma_i V_i(rA-x_i)} \mid U = \{x_i\}\right]\right] \\
&= E\left[\Pi_i E\left[e^{tV_i(rA-x_i)} \mid U = \{x_i\}\right]\right] \quad \text{since } V_i \text{ is i.i.d.} \\
&= E\left[e^{\Sigma_i \log E\left[e^{tV_i(rA-x_i)} \mid U=\{x_i\}\right]}\right] \\
&= E\left[e^{\int \log E\left[e^{tV(rA-x)}\right] U(dx)}\right] \\
&= E\left[e^{U(\log E[e^{tV(rA-x)}])}\right].
\end{aligned}$$

The last equality holds since U is Poisson with intensity $\alpha > 0$. Also, the last equality makes sense since $\exp\{\alpha \int_{R^d} E[e^{tV(rA-x)} - 1]dx\}$ is finite for $t \in R$.

Thus,

$$M_r(t) = \exp\{\alpha \int_{R^d} E[e^{tV(rA-x)} - 1]dx\} \text{ is finite for } t \in R.$$

Next, we will consider the limit of the cumulant generating function as follows. That is to say,

$$M(t) = \lim_{r\to\infty} \frac{1}{r^d} \cdot \log M_r(t)$$

$$= \lim_{r\to\infty} \frac{\alpha}{r^d} \cdot \int_{R^d} E[e^{tV(rA-x)} - 1]dx$$

$$= \lim_{r\to\infty} \alpha \sum_{k=1}^{\infty} \frac{t^k}{k!} \cdot E \int_{R^d} \int_{R^d} \cdots$$

$$\int_{R^d} 1_{(rA-ry)}(x_1) \cdots 1_{(rA-ry)}(x_k) dy dV(x_1) \cdots dV(x_k)$$

$$= \alpha \sum_{k=1}^{\infty} \frac{t^k}{k!} \cdot E \int_{R^d} \int_{R^d} \cdots$$

$$\int_{R^d} 1_{(A-x(1)/r)}(y) \cdots 1_{(A-x(k)/r)}(y) dy dV(x_1) \cdots dV(x_k)$$

(denote $x(i) = x_i$)

$$= \alpha \sum_{k=1}^{\infty} \frac{t^k}{k!} \cdot E \int_{R^d} \int_{R^d} \cdots$$

$$\int_{R^d} |(A - x_1/r) \cdots (A - x_k/r)| dV(x_1) \cdots dV(x_k)$$

$$= \alpha |A| \sum_{k=1}^{\infty} \frac{t^k}{k!} \cdot E \int_{R^d} \cdots \int_{R^d} dV(x_1) \cdots dV(x_k).$$

The last equality holds by the dominated convergence theorem since

$$1_{(A-x_1/r)\cdots(A-x_k/r)} \le 1_{(A-x_1/r)}.$$

Note that $|(A - x_1/r) \cdots (A - x_k/r)| \le |(A - x_1/r)| = |A|$.
Therefore,

$$M(t) = \alpha |A| \sum_{k=1}^{\infty} \frac{t^k}{k!} \cdot E[V(R^d)]^k$$
$$= \alpha |A| E[e^{tV(R^d)} - 1]$$
$$= \alpha |A| [M_{V(R^d)}(t) - 1].$$

Thus, we get that $I(x) = \sup_{t\in R} \{tx + \alpha|A| - \alpha|A| M_{V(R^d)}(t)\}$.

Theorem 1 says that $X_r(A)/r^d$ satisfies the large deviation principle with rate function $I(x)$ given as above.

3.3. Doubly stochastic process

DEFINITION 7. Let Z be a stationary, associated random measure and let X_Z be a conditionally Poisson with intensity measure Z. That is, X_Z is a Z-conditionally Poisson point process with intensity measure Z. X_Z is called a doubly stochastic Poisson process.

The following hypotheses are assumed to hold for such an environment Z.

(3.5) (a) $C_r(t) = \frac{1}{r^d} \cdot \log E[e^{tZ_r(A)}]$ is finite for every $t \in R$,
where A is a bounded Borel subset of R^d.

(3.6) (b) $M_Z(t) = \lim_{r\to\infty} C_r(t)$ exists and is differentiable for every $t \in R$.

Now, for a doubly stochastic process X_Z (we denote X_Z by $W = X_Z$), define $M_{W(A)}(t)$,

$$M_{W(A)}(t) = \lim_{r\to\infty} \frac{1}{r^d} \cdot \log E[e^{tW_r(A)}] \text{ exists for every } t \in R,$$

and

$$I(x) = \sup_{t\in R}\{tx - M_{W(t)}\}.$$

Then, we have the following property.

THEOREM 8. *Let X_Z be a doubly stochastic process with environment Z. Let $W_r(A)$ be a random variable defined as above for a bounded Borel subset A of R^d. Under the above hypotheses (3.5) and (3.6), $W_r(A)/r^d$ satisfies the large deviation principle with rate function*

$$I(x) = \sup_{t\in R}\{tx - M_Z(H(t))\}, \text{ where } H(t) = e^t - 1.$$

Proof. First, let us calculate the moment generating function of $W_r(A)$. Since W is Poisson point process with intensity Z,

$$\begin{aligned} E[e^{tW_r(A)}] &= E\left[E\left[e^{tW_r(A)} \mid Z\right]\right] \\ &= E\left[e^{\int (e^f - 1)Z(dx)}\right], \text{ where } f = t1_{rA} \end{aligned}$$

$$= E\left[e^{Z_r(e^f - 1)}\right].$$

So,

$$\begin{aligned} M_W(t) &= \lim_{r\to\infty} \frac{1}{r^d} \cdot \log E\left[e^{Z_r(e^f - 1)}\right] \\ &= \lim_{r\to\infty} \frac{1}{r^d} \cdot \log E\left[e^{(e^t - 1)Z_r(A)}\right] \end{aligned}$$

Since $M_Z(t)$ exists for every $t \in R$ by the hypothesis (3.6),

$$M_W(t) = M_Z(e^t - 1) \text{ and } M_W(t) = M_Z(H(t)) = M_Z \circ H(t),$$

where $H(t) = e^t - 1$.

Hence

$$I(x) = \sup_{t\in R}\{tx - M_Z \circ H(t)\}.$$

Moreover, since $H(t)$ is differentiable for every $t \in R$ and also the composition function of two differentiable functions is differentiable, $M_W(t)$ is differentiable for every $t \in R$. Theorem 1 says that $W_r(A)/r^d$ satisfies the large deviation principle with rate function $I(x)$ given as above.

References

1. M. D. Donsker, and S. R. Varadhan, *Asymptotic evaluation of certain Markov process expectations for large time* III, Comm. Pure Appl. Math. **29** (1976), 389–461.
2. ———, *Asymptotic evaluation of certain Markov process expectations for large time* II, Comm. Pure Appl. Math. **28** (1975a), 1–47, (1975b) 279–301.
3. Ellis, R., *Entropy, Large deviations and statistical mechanics*, Berlin-Heidelberg New York, Springer, 1985.
4. ———, *Large deviations for a general class of dependent random vectors*, Ann. Probab. **12** (1984), 1–12.
5. J. Gärtner, *On large deviations from the invariant measure*, Theory Probab. Appl. **22** (1977), 24–39.
6. P. Groeneboom, *Large deviations and asymptotic efficiencies*, Mathematisch Centrum, Amsterdam, 1980.
7. O. Kallenberg, *Random measures*, Academic Press, New York, 1983.
8. O. E. Lanford, *Entropy and equilibrium states in classical statistical mechanics*, Lecture Notes Phys. vol. 20, Springer-Verlag, Berlin-Heidelberg-New York, 1973.

9. R. T. Rockafellar, *Convex analysis*, Princeton Univ. Press, Princeton, N.J., 1970.
10. D. W. Stroock, *An introduction to the theory of large deviations*, Springer-Verlag, Berlin-Heidelberg-New York-Tokyo, Springer, 1984.

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