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SURFACES IN A RIEMANNIAN MANIFOLD WITH A BOUNDED CURVATURE

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1. Introduction

Let M be a complete Riemannian manifold with the sectional curvature K_M. One of the central problems in Riemannian Geometry is the study of the metric structure of M in the case when K_M is bounded either above or below by a constant. When K_M is nonnegative (bounded below by zero), by the Toponogov splitting theorem([2], [7]), the universal covering $\tilde{M}$ of M can be written as the isometric product $\overline{M} \times \mathbb{R}^k$, where $\mathbb{R}^k$ has its standard flat metric. Therefore, in M we have a submanifold corresponding to $\mathbb{R}^k$, which is obviously totally geodesic. On the other hand, when K_M is nonpositive, it is shown in [4], [5] that the universal covering of M contains a flat totally geodesic submanifold determined by the fundamental group of M, and we have a similar conclusion as in the case of nonnegative curvature.

These theorems suggest that if M has a bounded curvature, a submanifold with extreme value of sectional curvature is totally geodesic in M when it is suitably constructed. In this paper, we will construct a 2-dimensional submanifold Σ of M when K_M is bounded, and show that the sectional curvature of M takes the extreme value over the surface Σ if and only if Σ is totally geodesic in M and locally isometric to a surface with constant curvature.

For the basic notation and tools we refer to [1], [3], [6].

2. Main Results

Let M be a complete Riemannian manifold with sectional curvature $K_M \geq c$ or $K_M \leq c$, where c is a constant. We will denote by $\langle\ ,\ \rangle$ the Riemannian metric on M. Let $\gamma : [a, b] \to M$ be a geodesic and $E(s)$

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be a parallel vector field along γ such that $\|\gamma'(s)\| = 1$, $\|E(s)\| = 1$, and $\langle \gamma'(s), E(s)\rangle = 0$. Define a smooth map $F : [a,b] \times [0,\infty) \to M$ by $F(s,t) = \exp_{\gamma(s)}(tE(s))$. Denote

$$\gamma_t(s) = F(s,t), \quad \sigma_s(t) = F(s,t),$$
$$V = F_*(\frac{\partial}{\partial s}) = \gamma_t'(s),$$
$$T = F_*(\frac{\partial}{\partial t}) = \sigma_s'(t).$$

Note that for each $s \in [a,b]$, $\sigma_s : [0,\infty) \to M$ is a geodesic, and hence F is in fact a variation through geodesics. Therefore the variational vector field $V(t)$ along each geodesic σ_s is a Jacobi-field. Let ∇ denote the Levi-Civita connection on M. Since ∇ is a symmetric connection and $[V,T] = F_*([\frac{\partial}{\partial s}, \frac{\partial}{\partial t}]) = 0$, we have $\nabla_T V = \nabla_V T$ whenever they are defined. Along $\gamma = \gamma_o$, We have $V(0) = \gamma'(s)$ and $T = E$. Therefore, $\nabla_T V = \nabla_V T = 0$ along γ because E is parallel. For each s, $V(t)$ is a Jacobi-field along σ_s such that $\langle V(0), \sigma_s'(0)\rangle = 0$. Then from the fact that the Jacobi-field $V(t)$ is a solution to a second order ordinary differential equation with initial conditions,

$$\langle V(0), \sigma_s'(0)\rangle = 0$$
$$\langle \nabla_T V(0), \sigma_s'(0)\rangle = 0,$$

it is not difficult to see that $\langle V(t), \sigma_s'(t)\rangle = 0$ for every t, and we conclude that V and T are perpendicular to each other whenever they are defined. We assume that for each $s \in [a,b]$, the geodesic σ_s has no conjugate points or focal points, and therefore F is an immersion. In particular, if $c > 0$, F is defined only for $t < \frac{\pi}{2\sqrt{c}}$ (Myers and Bonnet, see [1]). We denote by Σ the 2-dimensional immersed submanifold with the induced metric. This surface Σ is the object of our study. We will show that the sectional curvature of M takes the extreme value over the surface Σ if and only if Σ is totally geodesic as an immersed submanifold. By assumption, the sectional curvature of M is bounded either above or below by c. Hence the extreme value of the sectional curvature means that $K_M = c$ over Σ. In the following lemma, we will show what this means in terms of the curvature tensor. The curvature tensor R on M is defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

and $K_M(X,Y)$ means the sectional curvature of the plane spanned by X and Y.

LEMMA 2.1. *Let M and Σ be as above. Then the following statements are equivalent:*

(i) $K_M(T,V) = c$ *over* Σ.
(ii) $R(T,V)V = cT\|V\|^2$.
(iii) $R(V,T)T = cV$.

Proof. If either (ii) or (iii) is true, then clearly $K_M = c$ over Σ because $\langle T, V\rangle = 0$. Therefore, it suffices to show that (i) implies both (ii) and (iii). The argument is exactly same for (ii) and (iii), and we will only show that (i) implies (ii).

For each $p \in \Sigma$, $R(-,V)V : T_pM \to T_pM$ is a symmetric linear transformation because R is symmetric. Let $N \subset T_pM$ be the set of all vectors perpendicular to V and $A : N \to N$ be the restriction of $R(-,V)V$ to the subspace N. Since T_pM is a vector space isomorphic to $\mathbb{R}^{n+1}$ where $n+1$ is the dimension of M, we can view this map A as a symmetric linear transformation in $\mathbb{R}^n$. Define $f(W) = \langle A(W), W\rangle$, and $g(x_1, x_2, \dots, x_n) = {x_1}^2 + {x_2}^2 + \cdots + x_n^2$. Then $S^{n-1} = \{(x_1, x_2, \dots, x_n) \mid g(x_1, x_2, \dots, x_n) = 1\}$, and $f|S^{n-1}$ denotes f restricted to S^{n-1} which is the sectional curvature because W, V are orthonormal. Let $\mathbb{B} = \{e_1, e_2, \dots, e_n\}$ be an orthonormal basis for $\mathbb{R}^n$, $n \times n$ matrix (a_{ij}) be the matrix of A relative to $\mathbb{B}$, and let $W = \sum_{k=1}^n w_k e_k$. Then

$$\begin{aligned} f(W) &= \langle A(W), W\rangle \\ &= \langle \sum_{i=1}^n (\sum_{j=1}^n a_{ij} w_j) e_i, \sum_{k=1}^n w_k e_k \rangle \\ &= \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n a_{ij} w_j w_k \delta_i^k \end{aligned}$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} w_j w_i.$$

Since A is a nonzero symmetric $n \times n$ matrix, the above equation becomes

$$\sum_{i=1}^{n} a_{ii} w_i^2 + 2 \sum_{i>j} a_{ij} w_i w_j.$$

Then we can get

$$\nabla f(W) = 2 \sum_{i=1}^{n} (\sum_{j=1}^{n} a_{ij} w_j) e_i = 2A(W).$$

Since $f|S^{n-1}$ has a maximum or minimum at T, by the Lagrange multipliers, there exists a nonzero real number λ such that $\nabla f(T) = \lambda \nabla g(T)$. Therefore, we get $A(T) = \lambda T$ because $\nabla f(T) = 2A(T)$ and $\nabla g(T) = 2T$. And we can see that λ is equal to $c\|V\|^2$, because

$$\begin{aligned} \lambda &= f(T) = \langle A(T), T \rangle \\ &= \langle R(T,V)V, T \rangle = \frac{\langle R(T,V)V,T \rangle}{\|T \wedge V\|^2} \|V\|^2 \\ &= K_M(T,V)\|V\|^2 = c\|V\|^2. \end{aligned}$$

We already used the fact that V is a Jacobi-field along each σ_s in order to show that $\{V, T\}$ forms an orthogonal frame field over Σ. Every Jacobi-field satisfies a second order ordinary differential equation called the *Jacobi-equation*, of which the solutions are uniquely determined by the initial conditions. Using this fact, we can show that the vector field V must be of a special form in the case of extreme sectional curvature. In the following proposition, $P_s(t)$ is the parallel vector field along σ_s with $P_s(0) = \gamma'(s)$. Then, of course, we have $\|P_s(t)\| = 1$ for each s and t.

PROPOSITION 2.2. *Let M and Σ be as above. The sectional curvature $K_M(T,V) \equiv c$ if and only if $V(t) = \cos(\sqrt{c}\ t)P_s(t)$ for $c > 0$ and $V(t) = \cosh(\sqrt{-c}\ t)P_s(t)$ for $c \leq 0$.*

Proof. We will verify the statement only in the case when $c > 0$. For a nonpositive number c, the proof would be exactly same with the corresponding functions. By Lemma 2.1, it suffices to show that $V(t) = \cos(\sqrt{c}\,t)P_s(t)$ if and only if $R(V,T)T = cV$.

We first assume that $V(t) = \cos(\sqrt{c}\,t)P_s(t)$. Since P_s is parallel along σ_s, we have $\nabla_T P_s = 0$ and hence

$$\begin{aligned}\nabla_T\nabla_T V &= \frac{d^2}{dt^2}\cos(\sqrt{c}\,t)\ P_s \\ &= -c\cdot\cos(\sqrt{c}\,t)P_s \\ &= -cV.\end{aligned}$$

Since $V(t)$ is a Jacobi-field along σ_s, it satisfies the Jacobi-equation,

$$\nabla_T\nabla_T V + R(V,T)T = 0.$$

Therefore, we conclude

$$R(V,T)T = -\nabla_T\nabla_T V = cV.$$

Conversely, if $R(V,T)T = cV$, then we have

$$\nabla_T\nabla_T V + cV = 0$$

because we already know V is a Jacobi-field along each geodesic σ_s. Therefore, $V(t) = \cos(\sqrt{c}\,t)P_s(t)$ is the unique solution satisfying the initial conditions

$$\begin{cases} V(0) = \gamma'(s), \\ \nabla_T V(0) = 0. \end{cases}$$

In order to prove Σ is totally geodesic, we have to show that the second fundamental form II of Σ vanishes. Since $\{V,T\}$ forms an orthogonal system, it suffices to show that $\nabla_V V$, $\nabla_T T$, and $\nabla_T V$ are tangential to the surface Σ. The most difficult part is to show $\nabla_V V$ has only tangential component, which is proved in the following lemma.

LEMMA 3. *If $V(t) = \cos(\sqrt{c}\, t)P_s(t)$ or $\cosh(\sqrt{-c}\, t)P_s(t)$, then $\nabla_V V$ has only tangential component.*

Proof. Once again we will prove the statement only in the case when $c > 0$, and hence F is defined for $t < \frac{\pi}{2\sqrt{c}}$. Denote by P the vector field over Σ defined by $P_s(t)$ at the point (s,t). Then,

$$\begin{aligned}\nabla_V V &= \nabla_{\cos(\sqrt{c}\, t)\ P}\{\cos(\sqrt{c}\, t)\ P\}\\ &= \cos(\sqrt{c}\, t)\ \nabla_P\{\cos(\sqrt{c}\, t)\ P\}\\ &= \cos(\sqrt{c}\, t)\{P[\cos(\sqrt{c}\, t)]\ P + \cos(\sqrt{c}\, t)\nabla_P P\}\\ &= \cos^2(\sqrt{c}\, t)\ \nabla_P P.\end{aligned}$$

Hence it suffices to show that $\nabla_P P$ is tangent to Σ. Since the Lie-bracket has the property,

$$[fV, gW] = fg[V,W] + fV[g]W - gW[f]V$$

and $[T,V] = 0$, we have

$$\begin{aligned}0 &= [T, \cos(\sqrt{c}\, t)\ P]\\ &= \cos(\sqrt{c}\, t)\ [T,P] - \sqrt{c}\ \sin(\sqrt{c}\, t)\ P.\end{aligned}$$

From the fact that F is defined for $t < \frac{\pi}{2\sqrt{c}}$, we know $\cos(\sqrt{c}\, t) \neq 0$ and hence

$$[T,P] = \sqrt{c}\ \tan(\sqrt{c}\, t)\ P.$$

Using this expression for Lie-bracket $[T,P]$, we can show that the vector field $\nabla_P P$ satisfies a first order differential equation along each σ_s. By the definition of the curvature tensor and lemma 2.1, we get

$$\nabla_T\nabla_P P - \nabla_P\nabla_T P - \nabla_{[T,P]}P = R(T,P)P = cT.$$

Since P is parallel along σ_s, we know $\nabla_T P = 0$. Together with $\nabla_{[T,P]}P = \sqrt{c}\ \tan(\sqrt{c}\, t)\nabla_P P$, we obtain

$$\nabla_T\nabla_P P - \sqrt{c}\ \tan(\sqrt{c}\, t)\nabla_P P = cT.$$

Put $W = \nabla_P P$. Then the above equation becomes

$$\nabla_T W - \sqrt{c}\ \tan(\sqrt{c}\, t)W = cT.$$

Take a parallel orthonormal frame field $\{P_i(t)\}_{i=1}^n$ along $\sigma_s(t)$ with $P_1(t) = T$. Then we can write

$$W(t) = \nabla_P P = \sum_{i=1}^n f_i(t)P_i(t),$$

and the above equation becomes

$$\sum_{i=1}^n f_i' P_i - \sqrt{c}\ \tan(\sqrt{c}\, t)\sum_{i=1}^n f_i P_i = cP_1.$$

Since γ_t is geodesic at $t = 0$, we have

$$W(0) = \nabla_{\gamma'(s)}\gamma'(s) = \sum_{i=1}^n f_i(0)P_i(0) = 0.$$

We get the initial condition $f_i(0) = 0$ for $1 \le i \le n$. Thus we get a system of first order ordinary differential equations,

$$\begin{cases} f_1' - \sqrt{c}\ \tan(\sqrt{c}\, t)f_1 = c, \\ f_i' - \sqrt{c}\ \tan(\sqrt{c}\, t)f_i = 0, \quad \text{for} \quad 2 \le i \le n \end{cases}$$

with the initial condition $f_i(0) = 0$ for $1 \le i \le n$.

The solutions to this system are

$$\begin{cases} f_1(t) = \sqrt{c}\ \tan(\sqrt{c}\, t), \\ f_i(t) \equiv 0 \text{ for } 2 \le i \le n. \end{cases}$$

Therefore, we have

$$\begin{aligned} W = \sum_{i=1}^n f_i P_i &= \sqrt{c}\ \tan(\sqrt{c}\, t)P_1 \\ &= \sqrt{c}\ \tan(\sqrt{c}\, t)T. \end{aligned}$$

Therefore $\nabla_V V$ has only tangential component.

We are now ready to prove our main theorem. By $\mathbb{S}(c)$ we denote the 2-dimensional rank one simply connected symmetric space of constant curvature c, which means $\mathbb{S}(c)$ is a sphere if $c > 0$, the Euclidean plane if $c = 0$, and a hyperbolic space if $c < 0$.

THEOREM 2.4. *Let M be a complete Riemannian manifold with the sectional curvature K_M either bounded above or below by c, where c is a constant. Then $K_M = c$ over Σ if and only if Σ is locally isometric to $\mathbb{S}(c)$ and totally geodesic.*

Proof. If Σ is totally geodesic and locally isometric to $\mathbb{S}(c)$, then $K_M = c$ over Σ by the Gauss formula.

If $K_M = c$ over Σ, then by lemma 2.1 Σ is locally isometric to $\mathbb{S}(c)$. The second fundamental form $\Pi(T,T) = (\nabla_T T)^\perp =$ the normal component of $(\nabla_T T) = 0$ because σ_s is geodesic. Furthermore,

$$\begin{aligned}\nabla_T V &= \nabla_T \cos\sqrt{c}\ t\ P\\ &= (-\sqrt{c}\sin\sqrt{c}\ t)P,\end{aligned}$$

which is tangent to Σ. Therefore $\Pi(T,V) = (\nabla_T V)^\perp = 0$ and by lemma 2.3, $\Pi(V,V) = (\nabla_V V)^\perp = 0$. Thus the second fundamental form Π is identically zero, that is Σ is totally geodesic.

COROLLARY 2.5. *Suppose that $K_M \le c$. If $K_\Sigma = c$, then Σ is totally geodesic in M.*

Proof. If $K_\Sigma = c$ then, by the Gauss formula,

$$\begin{aligned}c &= K_\Sigma\\ &= K_M - \frac{\|\Pi(T,V)\|^2}{\|T\wedge V\|^2}\\ &\le c.\end{aligned}$$

Equality holds only when $K_M = c$ over Σ and $\Pi(T,V) = 0$. Thus, by theorem 2.4, the corollary 2.5 is proved.

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