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ON THE INDEX THEOREM OVER EVEN–DIMENSIONAL CLOSED ORIENTED RIEMANNIAN MANIFOLDS

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Let M be a closed oriented Riemannian manifold with even-dimension throughout this note. We put

$$B(M) = \{x \in T(M) \mid \|x\| \leq 1\}$$

and

$$S(M) = \{x \in T(M) \mid \|x\| = 1\}$$

where $T(M)$ is the tangent bundle of M. Let

$$\pi : B(M) \to M$$

be the projection map. For a vector bundle ζ over M we define the following:

(1) $C^\infty(\zeta)$ is the set of all C^∞ cross sections $M \to \zeta$

(2) $\tilde{\zeta}$ is the restriction of $\pi^*\zeta$ to $S(M)$.

For two vector bundles ζ and η over M, let

$$D : C^\infty(\zeta) \to C^\infty(\eta)$$

be an elliptic operator with the symbol $\sigma(D)$, then we have the isomorphism

$$\sigma(D) : \tilde{\zeta} \cong \tilde{\eta}.$$

Hence we have an element $\gamma(D) \in K(B(M), S(M))$ (relative K-group) ([1], [2], [3]) such that

$$\gamma(D) = (\pi^*\zeta,\ \pi^*\eta,\ \sigma(D)).$$

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Moreover, we have two homomorphisms $i_a : K(B(M), S(M)) \to \mathbf{Z}$ (=integers) and $i_t : K(B(M), S(M)) \to \mathbf{Q}$ (=rationals) as follows. For $\gamma(D) \in K(B(M), S(M))$, $i_a(\gamma(D)) = i_a(D)$ and $i_t(\gamma(D)) = i_t(D)$, where $i_a(D)$ is the analytical index of D and $i_t(D)$ the topological index of D ([4], [5], [7]). The Atiyah- Singer index theorem asserts that $i_a(\gamma(D)) = i_t(\gamma(D))$ ([3], [4], [7]).

Let $T^*(M)$ be the cotangent bundle of M. We put

$$\xi^k = \Lambda^k(T^*(M) \otimes_R \mathbf{C}) \text{ and } \xi = \bigoplus_k \xi^k$$

where $\mathbf{C}$ is the ring of complex numbers. For the exterior derivative $d: C^\infty(\xi) \to C^\infty(\xi)$ let δ be the adjoint operator. Then, $d + \delta$ is an elliptic differential operator ([2], [3], [7]).

A bundle map $\alpha : \xi \to \xi$ is defined as follows. For each $k \geq 0$, $\alpha : \xi^k \to \xi^{2l-k}$ is defined by

$$\alpha = (i)^{-k(k-1)+l} * \quad (i = \sqrt{-1})$$

where $\dim_R(M) = 2l$ and $*$ is the star operator. Since $\alpha^2 = 1_\xi$ ([2], [3], [6]), we shall put

$$\xi^+ = \{v \in \xi \mid \alpha(v) = v\}, \quad \xi^- = \{v \in \xi \mid \alpha(v) = -v\},$$

then it is clear that

$$\xi = \xi^+ \oplus \xi^-$$

([3], [7]). We put

$$d + \delta|_{C^\infty(\xi^+)} = D_M : C^\infty(\xi^+) \to C^\infty(\xi^-).$$

Since $K(B(M), S(M))$ is a $K(M)$-module under the action:

$$\nu \cdot \beta = \pi^*\nu \otimes \beta \ (\nu \in K(M), \beta \in K(B(M), S(M)))$$

([1]). We compute $(D_M \otimes 1_\eta)$ as follows, where η is a vector bundle over M.

$$\begin{aligned}(D_M \otimes 1_\eta) &= d(\pi^*(\xi^+ \otimes \eta), \pi^*(\xi^- \otimes \eta), \sigma(D_M) \otimes 1_\eta) \\ &= d(\pi^*\xi^+, \pi^*\xi^-, \sigma(D_M)) \cup \pi^*\eta,\end{aligned}$$

where $\cup$ is the cup product.

Therefore we have the following:

$$(A) \qquad (D_M \otimes 1_\eta) = [\eta] \cdot \gamma(D_M).$$

We review the Pontrjagin classes of M as the elementary symmetric functions of $y_1^2, \ldots, y_l^2$, in the usual way:

$$1 + p_1(M) + \cdots + p_l(M) = \prod_{i=1}^{l} (1 + y_i^2).$$

Then

$$(B) \qquad \begin{aligned} ch(D_M) &= \prod_{i=1}^{l} (e^{y_i} - e^{-y_i})/y_i \\ &= 2^l \; + \; \text{positive dimensional terms} \end{aligned}$$

([3], [7]). In this case, an immediate consequence is

$$(C) \qquad ch(D_M \otimes 1_\eta) = ch(D_M) \cdot ch(\eta)$$

where $ch(\eta)$ is the Chern character of η([3], [7]). We define

$$i \; : \; K(M) \; \to \; \mathbf{Q}$$

by $i(\nu) = i(\nu, D_M)$ for each $\nu \in K(M)$. Thus from (A) we have

$$(A') \qquad i([\eta]) = i(D_M \otimes 1_\eta).$$

For each $\nu \in K(M)$ we use the notation $i(M, \nu)$ in place of $i(\nu)$ sometimes.

In this note, we want to prove that $i(S^{2m}, V) = 2^m$ where S^{2m} is the $2m$-dimensional sphere and $V \in K(S^{2m})$ such that $ch(V)$ is the generator of $H^{2m}(S^{2m}; \mathbf{Q})$ (Theorem 4).

PROPOSITION 1. *For the m-dimensional sphere, let $p_i(S^m)$ be the i^{th} Pontrjagin class of S^m. Then $p_i(S^m) = 0$ for all $i = 1, 2, 3, \ldots$.*

Proof. For the tangent bundle $T(S^m)$ of S^m

$$p_j(T(S^m)) = p_j(S^m) \in H^{4j}(S^m; \mathbf{Z}) \quad (j = 0, 1, 2, \ldots).$$

Since

$$T(S^m) \oplus \theta^1 = T(\mathbf{R}^{m+1})|_{S^m}$$

by the properties of Pontrjagin classes ([6]), we have

$$p(S^m) = p(T(S^m) \oplus \theta^1) = p(T(\mathbf{R}^{m+1})|_{S^m}) = 1,$$

where θ^1 is the one-dimensional vector bundle which is trivial and $p(S^m)$ is the total Pontrjagin class of S^m. Thus, from

$$1 = p_0(S^m) + p_1(S^m) + \cdots \quad \text{and} \quad p_0(S^m) = 1$$

([6]), we have

$$p_i(S^m) = 0 \quad \text{for} \quad i = 1, 2, \ldots.$$

LEMMA 2. *The index of the operator*

$$D_0 = d + \delta|_{C^\infty(\xi^e)} : C^\infty(\xi^e) \to C^\infty(\xi^0)$$

is the Euler characteristic of M, where

$$\xi^e = \bigoplus_k \xi^{2k}, \quad \xi^0 = \bigoplus_k \xi^{2k+1}.$$

Sketch of Proof. We put $\Delta = (d + \delta)^2$, then there is a canonical isomorphism $H^*(M; \mathbf{C}) \cong \ker \Delta$ by the Hodge theory ([3], [7]). For each $\beta \in \ker \Delta$ let φ be a representative of β which is called a harmonic representative (form) of β. In this case,

$$(*) \qquad \dim \ker(D_0) = \sum_{i=0}^{[m/2]} b_{2i}, \quad \dim \ker(D_0^*) = \sum_{i=0}^{[m/2]} b_{2i+1}$$

where D_0^* is the adjoint operator of D_0,

$$\sum_{i=0}^{m}(-1)^i b_i \quad (b_i = i^{th} \quad \text{Betti number of } M)$$

is the Euler characteristic of M and $\dim_R(M) = 2m$. By the Hodge theory ([7]), even-dimensional harmonic forms are 1-1 correspondence with even-dimensional elements of $H^*(M; \mathbf{C})$. The first half of (*) is proved. Since $D_0 = d + \delta$ is formally self-adjoint the second half of (*) follows from the argument of the preceding paragraph. Thus we have

$$i_a(D_0) = \sum_{i=0}^{m}(-1)^i b_i \quad (\text{=the analytic index of } D_0).$$

On the other hand, for the topological index $i_t(D_0)$ of D_0 we have

$$i_t(D_0) = e(M)([M]),$$

where $e(M)$ is the Euler class of M and $[M]$ the fundamental homology class of M. Since

$$e(M)([M]) = \sum_{i=0}^{m}(-1)^i b_i \quad (\text{Euler charateristic}) .$$

([6]) we have

$$i_a(D_0) = \sum_{i=0}^{m}(-1)^i b_i = i_t(D_0).$$

Let ζ be a complex vector bundle over M and let $c_i(\zeta)$ be the i^{th} Chern class of ζ. As is well-known, there are $x_i \in H^2(M; \mathbf{Q})$ such that

$$1 + c_1(\zeta) + \cdots + c_n(\zeta) = \prod_{i=1}^{n}(1 + x_i)$$

where $\dim_{\mathbf{C}} \zeta = n$ ([3], [7]). In this case,

$$\mathfrak{T}(\zeta) = \prod_{i=1}^{n} x_i/(1 - e^{-x_i})$$

is the *Todd class* of ζ ([3], [5], [7]).

Let ζ and η be two complex bundles over M, and let

$$D : C^\infty(\zeta) \to C^\infty(\eta)$$

be an elliptic differential operator. Then the topological index $i_t(D)$ is defined as follows:

$$i_t(D) = (ch(D)\mathfrak{T}(M))([M]),$$

where $ch(D)$ is the Chern character of $D, \mathfrak{T}(M)$ the Todd class of the tangent bundle ξ of M and $[M]$ the fundamental homology class of M.

PROPOSITION 3. (i) *If the Pontrjagin classes of M are zero, then $\mathfrak{T}(M) = \mathfrak{T}(\xi) = 1$.*

(ii) *For the $2m$-dimensional sphere S^{2m} let $\sigma \in H^{2m}(S^{2m}; \mathbf{Q})$ be the generator determined by the orientation: $\sigma([S^{2m}]) = 1$, where $[S^{2m}]$ is the fundamental homology class of S^{2m}. Then $ch(D_0) = 2\sigma$ (for D_0 see Lemma).*

Proof. (i) If the Pontrjagin classes are zero, in the definition of Todd class

$$\mathfrak{T}(M) = \prod_{i=1}^{m} x_i/(1-e^{-x_i}), \qquad \dim_{\mathbf{R}} M = 2m$$

$$x_i/(1-e^{-x_i}) = x_i/(1 - x_i + x_i^2/2! + \cdots) = 1$$

because

$$1 + p_1(M) + \cdots p_m(M) = \prod_{i=1}^{m}(1+x_i^2) = 1$$

implies $x_i = 0$ for all $i = 1, 2, \cdots$. Thus we have $\mathfrak{T}(M) = 1$.

(ii) By the Proposition 1, we see that the Pontrjagin classes of S^{2m} are zero. Therefore we have $\mathfrak{T}(S^{2m}) = 1$. From

$$\begin{aligned} i_t(D_0) &= (ch(D_0)\mathfrak{T}(S^{2m}))([S^{2m}]) \\ &= e(S^{2m})([S^{2m}]) \end{aligned}$$

we have

$$(ch(D_0)\mathfrak{T}(S^{2m})) = e(S^{2m}),$$

where $e(S^{2m})$ is the Euler class of S^{2m}. Thus

$$(ch(D_0)) = e(S^{2m})$$

by (i).

Let $\mathcal{D}(S^{2m})$ be the primary obstruction class of the tangent bundle of S^{2m}. We have

$$e(S^{2m}) = \mathcal{D}(S^{2m}) = 2\sigma$$

([6]). Thus

$$ch(D_0) = 2\sigma.$$

THEOREM 4. *For the 2m-dimensional sphere* $S^{2m}, i(S^{2m}, V) = 2^m$, *where* $V \in K(S^{2m})$ *is determined by* $ch(V) = \sigma$ *and* $\sigma \in H^{2m}(S^{2m}; \mathbf{Q})$ *such that* $\sigma([S^{2m}]) = 1$.

Proof. By (B) above

$$ch(D_{S^{2m}}) = 2^m$$

since the Pontrjagin classes of S^{2m} are zero (by Proposition 1).

Let M be a closed oriented Riemannian manifold with $\dim_{\mathbf{R}} M = 2m$, and let $B(M), S(M)$ be defined as follows:

$$B(M) = \{v \in T^*(M) | \; \|v\| \;\leq\; 1\}$$

$$S(M) = \{v \in T^*(M) | \; \|v\| \;=\; 1\},$$

where $T^*(M)$ is the cotangent bundle of M. Then we have the Thom isomorphism ([3], [6], [7])

$$\varphi_* : H^l(M, \mathbf{Q}) \to H^{2m+l}(B(M), S(M); \mathbf{Q})$$

defined by $\varphi_*(\alpha) = \pi^*\alpha \cup U$ for any $\alpha \in H^i(M, \mathbf{Q})$, where $\pi : T^*(M) \to M$ is the projection, $\cup$ the cup product and U is the Thom class in $H^{2m}(B(M), S(M); \mathbf{Q})$ satisfying the following:

For each $x \in M$ let $T^*(M)_x$ be the fiber of $T^*(M)$ at x. Then H^{2m} $(B(T^*(M)_x)$, $S(T^*(M)_x)$; $\mathbf{Q})(\cong \mathbf{Q})$ has the generator U_x. Then $j_x^*(U) = U_x$ where j_x is the inclusion

$$j_x : (B(T^*(M)_x), S(T^*(M)_x)) \to (B(M), S(M)).$$

With the Chern character ch and the isomorphism φ_* above we define the following

$$ch(D_\alpha) = (-1)^m \varphi_*^+(ch(\alpha))$$

where $\alpha = \gamma(D_\alpha) \in K(B(M), S(M))$ and D_α is an elliptic differential operator. Thus we have

$$(**) \qquad ch(\gamma(D_{S^{2m}})) = (-1)^m 2^m \cup U$$

where U is the Thom class in $H^{2m}(B(S^{2m}),\ S(S^{2m}); \mathbf{Z})$. By (A) and (C) above

$$(***) \qquad \begin{aligned} ch(\gamma(D_{S^{2m}} \otimes 1_V)) &= ch(\gamma(D_{S^{2m}})) \cdot ch(1_V) \\ &= (-1)^m 2^m \pi^* \sigma \cup U. \end{aligned}$$

On the other hand, for $\gamma(D_0) \in K(B(S^{2m}), S(S^{2m}))$

$$ch(\gamma(D_0)) = (-1)^m 2\pi^* \sigma \cup U$$

by (ii) of Proposition (3) and the isomorphism φ_*. In general, for a finite CW pair (X, Y), since

$$ch : K(X,Y) \otimes \mathbf{Q} \xrightarrow{\cong} H^{ev}(X, Y; \mathbf{Q})$$

the kernel of ch is the torsion subgroup of $K(X,Y)$ ([3], [7]), where

$$H^{ev}(X,Y;\mathbf{Q}) = \bigoplus_k H^{2k}(X,Y;\mathbf{Q}).$$

Hence, from $(**)$ and $(***)$ we have

$$(****) \qquad \gamma(D_{S^{2m}} \otimes 1_V) = 2^{m-1}\gamma(D_0) + \text{ torsion elements.}$$

By Lemma 2 and the proof of (ii) of Proposition 3, the Euler characteristic is $2 = e(S^{2m})([S^{2m}])$, and thus, as in the proof of Lemma 2

$$i_a(\gamma(D_0)) = i_t(\gamma(D_0)) = i_t(D_0) = 2.$$

Thus we have the following :

$$\begin{aligned} i(S^{2m}, V) &= i(D_{S^{2m}} \otimes 1_V) && (by (A')) \\ &= i(\gamma(D_{S^{2m}} \otimes 1_V)) \\ &= 2^{m-1} i(\gamma(D_0)) && (by (****)) \\ &= 2^m. \end{aligned}$$

References

1. M. Karoubi, *K-Theory*, Springer-Verlag, Berlin-Heidelberg-New York, 1978.
2. H. Lee, W Jeon and K. Shon, *A note on analytical indices, to appear.*
3. K. Lee and H. Lee, *Sheaves, complex manifolds and index theorem*, Hyongseol-Sa, Seoul, 1984.
4. K. Lee and H. Lee, *Some properties of analytical indices of* $K(B(M), S(M))$, Honam Math. J. **14** (1992), 1–6.
5. H. Lee and K. Sohn, *A note on topological indices*, Comm. Korean Math. Soc. **7** (1992), 25–32.
6. K. Lee, *Foundations of topology* II, Hakmoon-Sa, Seoul, 1984.
7. R. S. Palais, *Seminar on the Atiyah-Singer index theorem*, Princeton Univ. Press, Princeton, 1965.

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