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A CHARACTERIZATION OF TOTALLY UMBILIC SUBMANIFOLDS

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1. Introduction

A totally umbilic submanifold M^n of a Riemannian manifold $\overline{M}^{n+k}$ is a submanifold whose first fundamental form and second fundamental form are proportional. An ordinary hypersphere $S^n(r)$ of an affine $(n+1)$-space of $\mathbf{R}^{n+k}$ is the best known example of totally umbilic submanifolds of $\mathbf{R}^{n+k}$. From the point of views in differential geometry, the totally umbilic submanifolds are the simplest submanifolds next to totally geodesic submanifolds. The totally umbilic submanifolds of a space form $\overline{M}^{n+k}(c)$ with constant sectional curvature c are well known ([2] p. 129). Moreover the totally umbilic submanifolds of a locally symmetric space were classified in a series of papers by B.Y. Chen (see [1] for the references).

In this paper we give a characterization of totally umbilic submanifolds M^n of Riemannian manifold $\overline{M}^{n+k}$ and a characterization of totally umbilic spacelike submanifolds M^n of a semi-Riemannian manifold $\overline{M}_k^{n+k}$ of index k. In proving our results, we have used the method in [3].

2. Preliminaries

Let M^n be a submanifold of a Riemannian manifolds $\overline{M}^{n+k}$. We denote by σ, H and A_V the second fundamental form, the mean curvature vector field $(1/n)\mathrm{tr}(\sigma)$ and the shape operator with respect to a normal vector V. If we denote the covariant differentiation of the Riemannian

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manifold M^n and $\overline{M}^{n+k}$ by ∇ and $\overline{\nabla}$ respectively, then they are related by

$$\overline{\nabla}_X Y = \nabla_X Y + \sigma(X,Y). \tag{2.1}$$

And the second fundamental form and the shape operator are related by

$$\langle \sigma(X,Y), V\rangle = \langle A_V(X), Y\rangle. \tag{2.2}$$

Given a local orthonormal frame $\{e_1, \dots, e_{n+k}\}$ such that $e_1, \dots, e_n$ are tangent to M^n and $e_{n+1}, \dots, e_{n+k}$ are normal to M^n, we write $A_\alpha = A_{e_\alpha} (\alpha = n+1, \dots, n+k)$. In the sequel, i and j run over the range $\{1, \dots, n\}$ and α and β run over the range $\{n+1, \dots, n+k\}$. If we denote curvature tensors for the connections ∇, $\overline{\nabla}$ by $R, \bar{R}$ respectively, then we have the following structure equation of Gauss:

$$\begin{aligned}\langle R(X,Y)Z, W\rangle = \langle \bar{R}(X,Y)Z, W\rangle \\ + \langle \sigma(X,W), \sigma(Y,Z)\rangle - \langle \sigma(X,Z), \sigma(Y,W)\rangle.\end{aligned} \tag{2.3}$$

The Ricci curvature tensor of M^n and $\overline{M}^{n+k}$ are denoted by Ric and $\overline{Ric}$ respectively. By the equations (2.2), (2.3), we have

$$\begin{aligned}Ric(X,Y) = \overline{Ric}|_{TM}(X,Y) + \langle \sigma(X,Y), nH\rangle \\ - \sum_\alpha \langle A_\alpha(X), A_\alpha(Y)\rangle,\end{aligned} \tag{2.4}$$

where we denote $\sum_{i=1}^n \langle \bar{R}(e_i, X)Y, e_i\rangle$ by $\overline{Ric}|_{TM}(X,Y)$.

3. Main Theorems

Recall that totally umbilic submanifolds are submanifolds satisfying

$$\sigma(X,Y) = \langle X,Y\rangle H \tag{3.1}$$

for all $X, Y \in TM$.

PROPOSITION 3.1. *If M^n is a totally umbilic submanifold of $\overline{M}^{n+k}$, then for any unit tangent vector X to M^n,*

$$Ric(X,X) = \overline{Ric}|_{TM}(X,X) + (n-1)|\sigma(X,X)|^2. \tag{3.2}$$

Proof. Let M^n be a totally umbilic submanifold of $\overline{M}^{n+k}$. Then by the equations (2.2), (2.4) and (3.1), we have

$$\begin{aligned} Ric(X,X) - \overline{Ric}|_{TM}(X,X) &= \langle \sigma(X,X), nH\rangle \\ &\quad - \sum_\alpha \langle A_\alpha(X), A_\alpha(X)\rangle \\ &= n|X|^2|H|^2 - |X|^2|H|^2 \\ &= (n-1)|X|^2|H|^2 \\ &= (n-1)|\sigma(X,X)|^2, \end{aligned}$$

where the last equality follows from the fact that X is unit.

We will prove a converse of the above proposition as follows:

THEOREM 3.2. *Let M^n be a submanifold of $\overline{M}^{n+k}$ which satisfies*

$$Ric(X,X) \geq \overline{Ric}|_{TM}(X,X) + (n-1)|\sigma(X,X)|^2 \tag{3.3}$$

for all unit tangent vector X. Then M^n is a totally umbilic submanifold of $\overline{M}^{n+k}$.

Before proving the theorem, we give a lemma.

LEMMA 3.3. *Let M^n be a submanifold of $\overline{M}^{n+k}$ and p be a point in M^n. Then we have for all unit tangent vector $X \in T_pM$*

$$|\sigma(X,X)|^2 \leq \sum_\alpha |A_\alpha(X)|^2. \tag{3.4}$$

And equality holds for all unit tangent $X \in T_pM$ if and only if p is an umbilic point.

Proof. By the equation (2.2), we have

$$|\sigma(X,X)|^2 = \sum_\alpha \langle \sigma(X,X), e_\alpha\rangle^2 = \sum_\alpha \langle A_\alpha(X), X\rangle^2.$$

Hence the inequality (3.4) follows. And equality holds for all unit vector $X \in T_pM$ if and only if every tangent vector X at p is an eigenvector of A_α for any α. This means that p is an umbilic point.

Proof of the Theorem 3.2. Fix a point p in M^n and we denote $|\sigma(X,X)|^2$ by $h(X)$. Then h can be considered as a function from U_pM into R, where U_pM is the unit tangent sphere in T_pM and identified with S^{n-1} in $\mathbf{R}^n$. Let Y be a unit tangent vector of S^{n-1} at X, then a straightforward calculation as in [4] shows that at X we have

$$Yh = 4\langle\sigma(X,Y),\sigma(X,X)\rangle, \tag{3.5}$$

$$YYh = 4\{\langle\sigma(X,X),\sigma(Y,Y)\rangle - |\sigma(X,X)|^2 + 2|\sigma(X,Y)|^2\}. \tag{3.6}$$

Let $\{e_1,\cdots,e_{n-1}\}$ be an orthonormal basis of T_XS^{n-1}. Then $\{e_1,\cdots,e_{n-1},e_n = X\}$ forms an orthonormal basis of T_pM and by (3.6)

$$\frac{1}{4}e_ie_ih(X) = \langle\sigma(X,X),\sigma(e_i,e_i)\rangle - |\sigma(X,X)|^2 + 2|\sigma(X,e_i)|^2.$$

Hence we have

$$\begin{aligned}\frac{1}{4}\Delta_{S^{n-1}}h(X) &= \frac{1}{4}\sum_{i=1}^{n-1} e_ie_ih(X)\\ &= \langle\sigma(X,X),nH-\sigma(X,X)\rangle - (n-1)|\sigma(X,X)|^2\\ &\quad + 2\sum_{i=1}^{n-1}|\sigma(X,e_i)|^2.\end{aligned}$$

Since $\sum_{i=1}^{n-1}|\sigma(X,e_i)|^2 = \sum_\alpha |A_\alpha(X)|^2 - |\sigma(X,X)|^2$, we have

$$\frac{1}{4}\Delta_{S^{n-1}}h(X) = 2\sum_\alpha |A_\alpha(X)|^2 - (n+2)|\sigma(X,X)|^2 + \langle\sigma(X,X),nH\rangle.$$

Equation (2.4) gives a formula for the last term:

$$\langle\sigma(X,X),nH\rangle = Ric(X,X) - \overline{Ric}|_{TM}(X,X) + \sum_\alpha |A_\alpha(X)|^2.$$

Thus we have

$$\frac{1}{4}\Delta_{S^{n-1}} h(X) = \{Ric(X,X) - \overline{Ric}|_{TM}(X,X) - (n-1)|\sigma(X,X)|^2\} + 3\{\sum_{\alpha} |A_\alpha(X)|^2 - |\sigma(X,X)|^2\}.$$

Since we have

$$\int_{S^{n-1}} \Delta_{S^{n-1}} h(X) = 0,$$

the assumption (3.3) of the theorem implies that the second parenthesis in (3.7) must vanish for all X in T_pM. Hence by lemma 3.4 p is an umbilic point of M^n. Since p is an arbitrary point of M^n, we see that M^n is totally umbilic.

If $\overline{M}^{n+k}$ is the space form $\overline{M}^{n+k}(c)$ with constant sectional curvature c, then

$$\overline{Ric}|_{TM}(X,X) = (n-1)c|X|^2.$$

Thus we have

COROLLARY 3.4. *Let M^n be a submanifold of a space form $\overline{M}^{n+k}(c)$ which satisfies*

$$Ric(X,X) \geq (n-1)\{c + |\sigma(X,X)|^2\} \tag{3.8}$$

for all unit tangent vector X of M^n. Then M^n is totally umbilic.

If $\overline{M}_k^{n+k}$ is a semi-Riemannian manifold with index k and if M^n is a spacelike submanifold of $\overline{M}_k^{n+k}$, then we may prove the analogous theorems, which we state without proofs. We may find the basic definitions in [5].

PROPOSITION 3.5. *If M^n is a totally umbilic spacelike submanifold of a semi-Riemannian manifold $\overline{M}_k^{n+k}$, then for any unit tangent vector X to M^n,*

$$Ric(X,X) = \overline{Ric}|_{TM}(X,X) + (n-1)|\sigma(X,X)|^2.$$

THEOREM 3.6. *If M^n satisfies*

$$Ric(X,X) \leq \overline{Ric}|_{TM}(X,X) + (n-1)|\sigma(X,X)|^2$$

for all unit tangent vector X, then M^n is a totally umbilic submanifold of $\overline{M}_k^{n+k}$.

COROLLARY 3.7. *Let M^n be a spacelike submanifold of a semi-Riemannian manifold $\overline{M}_k^{n+k}(c)$ with index k and of constant sectional curvature c. If M^n satisfies*

$$Ric(X,X) \leq (n-1)\{c + |\sigma(X,X)|^2\}$$

for all unit tangent vector X of M^n, then M^n is totally umbilic.

REMARK. Since $\sigma(X, X)$ is orthogonal to M^n, $|\sigma(X, X)|^2 = \langle\sigma(X,X), \sigma(X,X)\rangle$ is nonpositive. And note that the inequality is reversed.

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