Comm. Korean Math. Soc. 8 (1993), No. 3, pp. 467–473

# WARPED PRODUCT SPACES WITH EINSTEIN METRIC

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## 0. Introduction

Warped products were first introduced by G. I. Kruǒkoviǒ in 1961, and then R. L. Bishop and B. O'Neill studied the manifolds with negative curvature under the name of warped products and they were used for instance by N. Ejiri, Y. Watanabe, A. Derdzinski and R. Palais to study conformal transformations and Einstein spaces. A surface of revolution is a typical example of warped products. The purpose of the present paper is to determine the warped product $\tilde{M} = B \times_f \overline{M}$ with Einstein metric and express the geometry of $\tilde{M}$ in terms of warping function $f$ and the geometries of $B$ and $\overline{M}$. In this paper we shall always deal with connected Riemannian manifolds with positive definite metric, and suppose that manifolds and quantities are differentiable of class $c^\infty$.

## 1. Preliminaries

Let $(B, g)$ and $(\overline{M}, \overline{g})$ be two Riemannian manifolds of dimensions $n$ and $p$ respectively, and let $f > 0$ be a smooth function on $B$. The warped product $\tilde{M} = B \times_f \overline{M}$ is the product manifold $B \times \overline{M}$ furnished with the metric tensor $\tilde{g} = \pi^*(g) + (f \circ \pi)^2 \sigma^*(\overline{g})$, where $\pi$ and $\sigma$ are the projections of $B \times \overline{M}$ onto $B$ and $\overline{M}$, respectively. If $f = 1$, then $B \times_f \overline{M}$ reduces to a Riemannian product manifold. $B$ is called the base of $\tilde{M}$, and $\overline{M}$ the fiber. The set of all lifts of $B$ and $\overline{M}$ being denoted by $\mathcal{L}(B)$ and $\mathcal{L}(\overline{M})$, respectively. We shall use the same notation for a vector field on $B$ or $\overline{M}$ and its lift on $\tilde{M}$. If the curvature tensor $R$ is

Received December 7, 1992.
This paper was supported by NON DIRECTED RESEARCH FUND, Korea Research Foundation, 1991.

defined by $R(E,F)G = D_E D_F G - D_F D_E G - D_{[E,F]}G$ for any vector fields $E, F$ and $G$ on $\tilde{M}$, then [1,3]

$$\tilde{g}(R(X,Y)Z,Z') = g(\hat{R}(X,Y)Z,Z'), \tag{1.1}$$

$$\tilde{g}(R(X,U)Y,V) = fDdf(X,Y)\overline{g}(U,V), \tag{1.2}$$

$$\begin{aligned}\tilde{g}(R(U,V)W,W') = f^2[&\overline{g}(\overline{R}(U,V)W,W') \\ &- \|df\|^2\{\overline{g}(U,W')\overline{g}(V,W) - \overline{g}(V,W')\overline{g}(U,W)\}],\end{aligned} \tag{1.3}$$

where $\hat{R}$ and $\overline{R}$ are the curvature tensors of $B$ and $\overline{M}$ respectively, and $Ddf$ is the Hessian of $f$ for $g$, $X,Y,Z,Z' \in \mathcal{L}(B)$ and $U,V,W,W' \in \mathcal{L}(\overline{M})$.

The components of Ricci tensors are given by

$$\tilde{S}(X,Y) = S(X,Y) - \frac{p}{f}Ddf(X,Y), \tag{1.4}$$

$$\tilde{S}(X,U) = 0, \tag{1.5}$$

$$\tilde{S}(U,V) = \overline{S}(U,V) - \overline{g}(U,V)[f\Delta f + (p-1)\|df\|^2], \tag{1.6}$$

where $\Delta f$ is the Laplacian of $f$ for $g$, and $\tilde{S}, S$ and $\overline{S}$ are the Ricci tensors of $\tilde{M}$, $B$ and $\overline{M}$ respectively.

Let $\tilde{k}, k$ and $\overline{k}$ be the scalar curvatures of $\tilde{M}, B$ and $\overline{M}$, then we have

$$\tilde{k} = k + f^{-2}\overline{k} - 2pf^{-1}\Delta f - p(p-1)f^{-2}\|df\|^2. \tag{1.7}$$

PROPOSITION 1 [1]. *The Warped product $\tilde{M} = B \times_f \overline{M}$ is Einstein with $\tilde{S} = \tilde{\lambda}\tilde{g}$ if and only if*

(1) *$\overline{M}$ is Einstein with $\overline{S} = \overline{\lambda}\overline{g}$,*
(2) *$\tilde{\lambda}g(X,Y) = S(X,Y) - pf^{-1}Ddf(X,Y)$,*
(3) *$\tilde{\lambda} = f^{-2}\overline{\lambda} - f^{-1}\Delta f - (p-1)f^{-2}\|df\|^2$.*

COROLLARY 2. *Let $\tilde{M} = B \times_f \overline{M}$ and $B$ be the Einstein spaces, then we have*

(1) *$\overline{M}$ is Einstein,*
(2) *$Ddf(X,Y) = \alpha f g(X,Y)$,*
(3) *$\tilde{\lambda} = f^{-2}\overline{\lambda} - f^{-1}\Delta f - (p-1)f^{-2}\|df\|^2$,*

*where we have put $\alpha = (\lambda - \tilde{\lambda})/p$, so that $\alpha$ is constant.*

## 2. Complete manifolds admitting a special concircular scalar field

Let $M$ be an $n$-dimensional Riemannian manifold with metric tensor $g_M$. We call a non-constant scalar field $\rho$ in $M$ a concircular scalar field if it satisfies the equation

$$D_X D_Y \rho = \phi g_M(X, Y), \tag{2.1}$$

where $D$ indicates covariant differentiation with respect to $g_M$ and $\phi$ is a scalar field, called the characteristic function of $\rho$. If $\phi$ is of the form $\phi = -\alpha\rho + \beta$ with constant coefficients $\alpha$ and $\beta$, then $\rho$ is called a special concircular scalar field. The term "concircular" comes form the concircular transformation introduced by K. Yano [5]. A concircular transformation is by definition a conformal transformation preserving geodesic circles.

We denote the number of isolated stationary points of a concircular scalar field $\rho$ in $M$ by $n(\rho)$. Y. Tashiro [4] studied the complete manifolds admitting concircular or special concircular scalar fields.

THEOREM 3 [4]. *Let $M$ be a complete Riemannian manifold of dimension $n \geq 2$ and suppose that it admits a special concircular field $\rho$ satisfying the equation*

$$D_X D_Y \rho = (-\alpha\rho + \beta) g(X, Y). \tag{2.2}$$

*Then $M$ is one of the following manifolds:*

- (I.A) *if $\alpha = \beta = 0$, the direct product $V \times I$ of an $(n-1)$-dimensional complete Riemannian manifold $V$ with a straight line $I$,*
- (I.B) *if $\alpha = 0$ but $\beta \neq 0$, a Euclidean space,*
- (II.A) *if $\alpha = -c^2 < 0$ and $n(\rho) = 0$, a pseudo-hyperbolic space of zero or negative type,*
- (II.B) *if $\alpha = -c^2 < 0$ and $n(\rho) = 1$, hyperbolic space of curvature $-c^2$ and*
- (III) *if $\alpha = c^2 > 0$, a spherical space of curvature $c^2$, where $c$ is a positive constant.*

## 3. Main Theorem

Let 0 be a point of a Riemannian manifold $M$ and $N$ a geodesic hypersphere with center 0. Take a normal coordinate system $(u, u^h)$ with center 0, where $u$ be the arc-length of any geodesic ray issuing from 0 and $u^h$ are coordinates of $N$, and let the metric form of $N$ be $dS_N^2 = g'_{ij} du^i du^j$. We shall first prove

THEOREM 4*. *If the metric form of $M$ is given by*

$$dS^2 = du^2 + h^2(u) dS_N^2 \tag{3.1}$$

*for some function $h(u)$ of $u$, with respect to normal coordinate system $(u, u^h)$ except at 0, then we obtain the following properties:*

(1) *The function $h(u)$ is differentiable at $u = 0$ and satisfies*

$$h(0) = 0, \quad h'(0) \neq 0. \tag{3.2}$$

(2) *The $(n-1)$-dimensional manifold $N$ with metric $g'$ is isometric to a sphere of curvature $k' = h'^2(0)$, provided $n - 1 \geq 2$.*

(3) *The manifold $M$ is of constant curvature $k$ if and only if $h$ satisfies the equation*

$$h'^2 + kh^2 = k', \tag{3.3}$$

*provided $n \geq 3$ and*

$$h'' + kh = 0, \tag{3.4}$$

*provided $n = 2$. The equation (3.4) is derived from (3.3).*

*Proof.* In the case of $n = 2$, the fact that $dS^2$ is non-degenerate at 0 is equivalent to that $h^2(u)/u^2$ has a non-zero limit as $u$ tends to 0 (see J. L. Kazdan and F. W. Warner [2] and A. Besse [1, p.96]). This implies the property (1).

In the case of $n > 2$, we consider a geodesic hypersphere $N$ with center 0, and let $X$ and $Y$ be vectors at 0. All geodesics tangent to the

*Professor Y. Tashiro kindly suggested me the proof of Theorem 4.

2-plane spanned by $X$ and $Y$ at 0 compose a surface $S$. The induced metric of the surface $S$ is given by

$$du^2 + h^2 dS_N^2|_{(N \cap S)},$$

where $|$ indicates the restriction on the intersection $N \cap S$. By the above argument in the case of $n = 2$, we have also the property (1).

(2) With respect to $(u, u^h)$ except at 0, the metric tensor $g$ of $M$ has components

$$g_{11} = 1,\ g_{i1} = 0,\ g_{ij} = h^2 g'_{ij},$$

the Christoffel symbol $\{{}_\lambda{}^\kappa{}_\mu\}$ of $M$ has components

$$\{{}_i{}^h{}_1\} = \frac{h'}{h}\delta_i^h,\ \{{}_i{}^1{}_j\} = -hh'g'_{ij},\ \{{}_i{}^h{}_j\} = \{{}_i{}^h{}_j\}_{g'}$$

and the others are zero, where $\{\ \}_{g'}$ is the Christoffel symbol composed from $g'$ of $N$. The curvature components $K_{\lambda\mu v}{}^\eta$ of $M$ has components

$$K_{j11}{}^h = -\frac{h''}{h}\delta_j^h, \quad K_{j1i}{}^1 = hh''g'_{ij}, \quad K_{jki}{}^h = K'_{jki}{}^h - h'^2(g'_{ki}\delta_j^h - g'_{ji}\delta_k^h),$$

their equivalents and the others are zero, where $K'_{kji}{}^h$ is the curvature tensor of $N$ with respect to $g'$. The square of the magnitude $\|K_{\lambda\mu v}{}^\kappa\|$ is equal to

$$\|K_{\lambda\mu v}{}^\kappa\|^2 = h^{-4}\|K'_{kji}{}^h - h'^2(g'_{ij}\delta_k^h - g'_{ik}\delta_j^h)\|_{g'}^2 + 4(n-1)h''^2 h^{-2},$$

where $\|\ \|_{g'}$ indicates the magnitude with respect to the metric $g'$. Since $h(u) \to 0$ as $u \to 0$ and $g'_{ij}$ and $K'_{kji}{}^h$ are independent of $u$, it follows that

$$K'_{jki}{}^h = h'^2(0)(g'_{ki}\delta_j^h - g'_{ji}\delta_k^h), \tag{3.6}$$

provided $n \geq 3$, that is $N$ with metric $g'$ is of constant curvature $k' = h'^2(0)$. As a geodesic hypersphere is diffeomorphic to an $(n-1)$-sphere, the manifold $N$ itself is isometric to an $(n-1)$-sphere of curvature $k'$.

(3) If $M$ is of constant curvature $k$, that is,

$$K_{\lambda\mu v}{}^{\eta} = k(g_{\mu v}\delta^{\eta}_{\lambda} - g_{\lambda v}\delta^{\eta}_{\mu}),$$

then comparing the component $K_{kji}{}^{h}$ with the third expression of (3.5), we have the equation (3.3). Deriving it in $u$, we have $h''/h = -k$ which satisfies the first and second equations of (3.5) for the constant sectional curvature of $M$. The converse is clear. In the case of $n = 2$, the Gaussian curvature of $M$ is given by

$$\frac{K_{1221}}{g_{22}} = -\frac{h''}{h}.$$

If this equals to a constant $k$, then we have the equation (3.4).

By the Corollary 2, we see that the base space $B$ of the warped product $M = B \times_f \overline{M}$ admits the special concircular scalar field $f$, that is, $Ddf = \gamma f g(X,Y)$ for $\gamma = (\tilde{\lambda} - \lambda)/p$. Hence, by virtue of Theorems 3 and 4, we can state.

THEOREM 5. *Let $\tilde{M} = B \times_f \overline{M}$ be a complete Einstein Riemannian manifold and $B$ be the Einstein one, then we have*

*(I) If $\gamma = 0$, then $\tilde{M} = (V \times I) \times_f \overline{M}$ of an $(n-1)$-dimensional complete Riemannian manifold $V$, straight line $I$ and an Einstein manifold $\overline{M}$, and the warping function $f$ is given by $f = ax^n$, where $a$ is arbitrary constant and $x^n$ is on $I$. Hence the metric form of $\tilde{M}$ is given by*

$$dS^2 = (dx^n)^2 + (ax^n)^2 d\overline{S}^2 + a^2 dS'^2$$

*for $d\overline{S}^2$ and $dS'^2$ are metric forms of $\overline{M}$ and $V$, respectively. Thus, if $\overline{M}$ is the geodesic hypersphere of $I \times_f \overline{M}$, then by use of Theorem 4, $\overline{M}$ is the sphere of curvature $a^2$ and $\tilde{M}$ is the product spaces of $(p+1)$-dimensional Euclidean space $I \times_f \overline{M}$ and $V$.*

*(II, A) If $\gamma = -c^2 < 0$ and $n(\rho) = 0$, then $\tilde{M}$ is the warped products of a pseudo-hyperbolic space of zero or negative type and Einstein manifold $\overline{M}$. Moreover the warping function $f$ is given by*

$$f = \begin{cases} (\mathrm{II}, A_0) & a \exp cx^n, \\ (\mathrm{II}, A_-) & a \sinh cx^n. \end{cases}$$

*For the case of* (*II*, $A_0$), $M$ *is the warped products of a pseudo-hyperbolic space. If* $f$ *is* (*II*, $A_-$) *and if* $\overline{M}$ *is the geodesic hypersphere of* $I \times_f \overline{M}$, *then* $\overline{M}$ *is the sphere of* $a^2c^2$ *provided* $p \geq 2$. *Thus* $\tilde{M}$ *is the product spaces of* $(p+1)$*-dimensional space* $I \times_f \overline{M}$ *of constant curvature* $K = -c^2 < 0$ and $V$ *if and only if* $K = -c^2$ *except at* $x^n = 0$.

(II, B) *If* $\gamma = -c^2 < 0$ *and* $n(\rho) = 1$, *then* $\tilde{M}$ *is the warped products of the hyperbolic space of curvature* $\alpha$ *and an Einstein space. Moreover the warping function* $f$ *is given by*

$$f = a \cosh cx^n.$$

(III) *If* $\gamma = c^2 > 0$, *then* $f$ *is given by* $f = a \cos cx^n$ and $\tilde{M}$ *is the warped products of the spherical space of curvature* $\gamma$ and an *Einstein space.*

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