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A CLASS OF GENUS ZERO MINIMAL SURFACES

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1. Introduction

We will consider conformal minimal immersions from a Riemann surface into R^3. It is an important theorem of Chern and Osserman that a complete minimal surface M in R^n is of finite total Gaussian curvature if and only if M is biholomorphic to a compact Riemann surface M_g punctured at a finite number of points and the tangential Gauss map extends holomorphically to all of M_g.

Jorge and Meeks [4] constructed an example of a complete minimal surface in R^3 with total curvature $-4\pi n$ (for each integer $n \geq 1$) conformally equivalent to

$$CP^1\backslash\{1, e^{2\pi i/(n+1)}, \dots, e^{2n\pi i/(n+1)}\}$$

whose ends are embedded.

In this article we carefully review the example of Jorge-Meeks [4], including what happens near the point at infinity. We will prove the following result.

PROPOSITION 4. *Suppose $f : M \to R^3$ is a complete conformal minimal immersion of finite total curvature. Also suppose that $\Psi : \widetilde{M} \to M$ is a biholomorphism, where $\widetilde{M}$ is another Riemann surface. Then $f \circ \Psi$ is a complete conformal minimal immersion of finite total curvature.*

We make the observation that there exists a Möbius transformation

$$\Psi : CP^1\backslash\sum_r \to CP^1\backslash\{e^{2(k-1)\pi i/k}, 1 \leq k \leq r\}$$

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where r is a positive integer less than 4, and $|\sum_r| = r$. Combining this observation with the aforementioned proposition we obtain the following immersion result: Given any subset $\sum_r \subset CP^1$ consisting of r points with $1 \leq r \leq 3$, there exists a complete conformal minimal immersion of finite total curvature $f : CP^1 \backslash \sum_r \to R^3$.

2. The Weierstrass Representation

In this section we brush up on the Weierstrass representation of a minimal surface in R^3. For proofs and details we refer the reader to [6]. Let M be a Riemann surface, and consider a conformal immersion

$$f = (f^i) : M \to R^3.$$

Conformality means that the induced metric $f^*ds_E^2$, ds_E^2 the Euclidean metric, is compatible with the complex structure of M in the following sense: If z is a local holomorphic coordinate, then the induced metric can be written as

$$ds^2 = h(z)\, dz \cdot d\bar{z}$$

for some $h(z) > 0$, where we write $f^*ds_E^2 = ds^2$. Writing $z = x + iy \in C$, we can rewrite the above as

$$ds^2 = h(x, y)(dx^2 + dy^2).$$

The local functions (x, y) are called isothermal coordinates.

The map f is said to be minimal if its mean curvature H vanishes identically. The mean curvature of f is related to the Laplacian of f as follows:

$$-2H = \Delta f,$$

where $\Delta = -(\frac{4}{h})\partial^2/\partial z\partial\bar{z}$. From the above formula we see that f is minimal if and only if $\Delta f = 0$, where Δ is the Laplacian of (M, ds^2). Hereafter $f : M \to R^3$ denotes a conformal minimal immersion from M. The minimality of f says that $\partial^2 f/\partial z\partial\bar{z} = 0$, and we put $\eta^i(z) = \partial f^i/\partial z$. Then we have $\partial\eta^i/\partial\bar{z} = 0$, i.e. η^i is a local holomorphic function on M. In fact each $\zeta^i = \eta^i(z)dz$ is a globally defined holomorphic 1-form on M by the chain rule. Therefore, given a conformal minimal immersion

$f: M \to R^3$ there arise three holomorphic 1-forms $\zeta_f = (\zeta^1, \zeta^2, \zeta^3)$ on M. Then it is not difficult to verify the following. Let f and ζ_f be as in the preceding discussion. Then we have

$$\sum |\eta^i|^2 = h/2, \text{ where } ds^2 = h dz \cdot d\bar{z};$$
$$\sum (\eta^i)^2 = 0;$$
$$(\zeta^i) \text{ have no real periods .}$$

Conversely, we have

PROPOSITION 1. *Let M be a Riemann surface. Suppose we have holomorphic 1-forms (ζ^i) on M satisfying*
(1) $\sum |\eta^i|^2 > 0$, *where $\zeta^i = \eta^i dz$ locally;*
(2) $\sum (\eta^i)^2 = 0$;
(3) (ζ^i) *have no real periods.*

Then $f = 2\Re e \int_{z_0}^{z} (\zeta^i) : M \to R^3$ is a conformal immersion with $f(z_0) = 0$.

For a proof of this well-known result see [6] *pp.* 15-16.

REMARK. By (3) we mean that the real part of the integral $\int_\gamma (\zeta^i)$ vanishes for any 1-cycle γ.

Let φ be a meromorphic function on a Riemann surface M, and also let μ be a not identically zero holomorphic 1-form on M. We further require that φ has a pole of order m at $p \in M$ if and only if μ has a zero of order $2m$ at p. Put

$$\zeta^1 = \frac{1}{2}(1-\varphi^2)\mu, \ \zeta^2 = \frac{i}{2}(1+\varphi^2)\mu, \ \zeta^3 = \varphi\mu.$$

The ζ^i's have no common zeros, hence the condition $\sum |\eta^i|^2 > 0$ is met. The condition $\sum (\eta^i)^2 = 0$ is also easily verified. Therefore the forms $\zeta = (\zeta^i)$ define a conformal minimal immersion $f = f_\zeta : M \to R^3$.

Put as before $\zeta^i = \eta^i dz \, (\eta^i = \partial f^i / \partial z)$. Assume that $\zeta^1 - i\zeta^2$ is not identically zero (when $\zeta^1 - i\zeta^2 \equiv 0$, f is a horizontal plane anyway). We put

$$\mu = \zeta^1 - i\zeta^2, \ \varphi = \zeta^3/\mu.$$

We then obtain a meromorphic function φ and a holomorphic 1-form μ on M satisfying the prescription that at a pole of φ of order m μ has a zero of order $2m$. To see this note that

$$(\eta^1 - i\eta^2)\cdot(\eta^1 + i\eta^2) = -(\eta^3)^2,\ \eta^1 + i\eta^2 = -(\frac{\mu}{dz}\cdot\varphi^2),$$

and $\eta^1 + i\eta^2$ is holomorphic. The pair $\{\mu, \varphi\}$ is called the Weierstrass pair of f.

The induced metric on M of a conformal minimal immersion $f : M \to R^3$ is given by, in terms of the Weierstrass pair,

$$ds^2 = (1 + |\varphi|^2)^2|\eta|^2 dz \cdot d\bar{z}, \quad \mu = \eta dz.$$

The holomorphic Gauss map of a conformal immersion $f : M \to R^3$ is defined to be the map

$$\Phi_f : M \to CP^2, \quad \Phi_f(z) = [\eta^1(z), \eta^2(z), \eta^3(z)],$$

where $[\eta^i] = [\partial f^i/\partial z]$.

PROPOSITION 2. *The map f is minimal if and only if Φ_f is holomorphic.*

Proof. Note that $\Delta f = 0$ if and only if $\partial(\partial f/\partial z)/\partial\bar{z} = 0$. And this is so if and only if $\partial\eta/\partial\bar{z} = 0$.

Let K denote the Gaussian curvature of the induced metric. Then $K \leq 0$, and

$$\tau_f = \int_M K dA \leq 0$$

is called the total curvature of f. The Gauss map of f, Φ_f, is said to be algebraic if the following holds:

(4) M is biholomorohic to a compact Riemann surface M_g punctured at a finite set of points $\sum_r = \{p_1, \cdots, p_r\}$;

(5) Φ_f extents holomorphically to all of M_g, which we again denote by Φ_f.

Suppose Φ_f is algebraic. Then the degree of Φ_f is, by definition, the degree of the algebraic curve $\Phi_f(M_g) \subset CP^2$.

The following result is a variant of the so called Wirtinger theorem from algebraic geometry, and a proof can be found in [6] pp. 24–25.

PROPOSITION 3. *Let τ_f denote the total curvature of f. Then*

$$-\tau_f = 2\pi deg(\Phi_f).$$

In particular, the total curvature is an integral multiple of 2π.

A Riemannian manifold (N, ds_N^2) is said to be complete if it is complete as a metric space. We have the following fundamental result.

The Chern-Osserman Theorem [3]. Suppose $f : M \to R^3$ is a complete minimal surface. Then the total curvature is finite if and only if the Gauss map is algebraic.

3. The Jorge-Meeks Surfaces

Let r denote the number of punctures, i.e., $r = |\sum_r|$. For each integer $r \geq 1$, Jorge and Meeks [4] constructed an example of a complete minimal surface in R^3 with total curvature $-4\pi(r-1)$, conformally equivalent to the sphere minus r points, whose ends are embedded.

We will review their construction for $r = 3$. Identify CP^1 with $C \cup \{\infty\}$, and set

$$M = CP^1 \backslash \{z \in C; z^3 = 1\},$$

i.e., $M = CP^1 \backslash \{1, e^{2\pi i/3}, e^{4\pi i/3}\}$. Put

$$p_1 = 1, p_2 = e^{2\pi i/3}, p_3 = e^{4\pi i/3}.$$

For $z \in CP^1$ we define

$$\varphi(z) = z^2.$$

In the neighborhood of infinity $CP^1 \backslash \{0\}$ we use the coordinate w, related to the Euclidean coordinate via $w = \frac{1}{z}$. Thus on $CP^1 \backslash \{0\}$

$$\varphi(z) = \varphi(\frac{1}{w}) = \frac{1}{w^2}.$$

Similarly for $z \in C \subset CP^1$ we define

$$\mu(z) = \frac{dz}{(z^3-1)^2} = \frac{dz}{(z-p_1)^2(z-p_2)^2(z-p_3)^2}.$$

On $CP^1\backslash\{0\}$,

$$\mu(z) = \mu(\frac{1}{w}) = \frac{-w^4 dw}{(1-w^3)^2}.$$

Then restrict φ and μ to M to obtain a Weierstrass pair on M.

$$\varphi(z) = \begin{cases} z^2 & \text{on } M\backslash\{\infty\}, \\ \frac{1}{w^2} & \text{on } M\backslash\{0\}, \end{cases}$$

$$\mu(z) = \begin{cases} \frac{dz}{(z^3-1)^2} & \text{on } M\backslash\{\infty\}, \\ \frac{-w^4 dw}{(1-w^3)^2} & \text{on } M\backslash\{0\}. \end{cases}$$

The function φ is meromorphic on M and μ is a holomorphic 1-form on M such that φ has a pole of order m at p_i and μ has a zero of $2m$ at p_i. Restricting to $M\backslash\{\infty\}$ we obtain holomorphic 1-forms

$$\zeta^1 = \frac{1-z^4}{2(z^3-1)^2}dz,$$

$$\zeta^2 = \frac{i(1+z^4)}{2(z^3-1)^2}dz,$$

$$\zeta^3 = \frac{z^2}{(z^3-1)^2}dz.$$

We want to show that the ζ^i's satisfy condition (1), (2), (3) of Proposition 1. For this, note that since φ has a pole of order 2 and μ has a zero of order 4 at the p_i's, Condition (1) and (2) hold. It remains to show that the ζ^i's have no real periods on M. Observe that the period condition is satisfied once

$$\Re e \int_{\gamma_a} \zeta^i = 0, \ 1 \le a, i \le 3,$$

where γ_a is a small circle about p_a. We will compute the period with respect to γ_3:

$$\int_{\gamma_3} \zeta^1 = 2\pi i \ Res_{p_3} \frac{1}{2}(\mu - \varphi^2\mu) = -\frac{2}{9}\pi i,$$

$$\int_{\gamma_3} \zeta^2 = 2\pi i \ Res_{p_3} \frac{i}{2}(\mu + \varphi^2\mu) = -\frac{2\sqrt{3}}{9}\pi i,$$

$$\int_{\gamma_3} \zeta^3 = 2\pi i \ Res_{p_3} \varphi\mu = 0$$

and we see that it has no real periods.

To take care of the point at infinity, we go to the neighborhood $M\backslash\{0\}$ and obtain holomorphic 1-forms

$$\zeta^1 = \frac{(w^4-1)dw}{2(1-w^3)^2},$$
$$\zeta^2 = \frac{-i(w^4+1)dw}{2(1-w^3)^2},$$
$$\zeta^3 = \frac{-w^2 dw}{(1-w^3)^2}.$$

In this case, we have a holomorphic 1-form μ on M and a meromorphic function φ on M such that μ has a zero of order 4 and φ has a pole of order 2 at $w=0$, and the conditions (1), (2) of Proposition 1 hold. The verification of condition (3) is similar to that given for $M\backslash\{\infty\}$. Finally, we will show that the arc-length of a curve approaching a puncture point p_a is infinite. Consider a divergent path

$$\gamma : [0, L) \to M,$$

parameterized by arc-length. We can then write

$$\gamma(s)^3 = 1 + r(s)\cdot e^{i\cdot\alpha(s)}.$$

Then $\lim_{s\to L} r(s) = 0$ and

$$3|\gamma|^2 = |e^{i\alpha(s)}|\cdot|r'(s) + i\cdot\alpha'(s)\cdot r(s)| \geq r'(s).$$

For s sufficiently close to L, $|\gamma|$ is close to 1, and $3|\gamma|^2 \leq A$ for some constant A. Now the arc length is given by

$$L(\gamma) = \int_\gamma \sqrt{h(z)}|dz|.$$

The metric on M is given by

$$ds^2 = h(z)dz\cdot d\bar{z},$$

$$h(z) = (1 + |\varphi|^2)^2|\eta|^2 = \frac{(1 + |z^2|^2)^2}{|(z^3 - 1)^2|^2}.$$

Therefore,

$$\begin{aligned}
L(\gamma) &= \int_\gamma \frac{1 + |z^2|^2}{|z^3 - 1|^2}|dz| \\
&= \int_\gamma \frac{1 + |\gamma(s)^2|^2}{r(s)^2}|\gamma'(s)|ds \ (|\gamma'(s)| = 1) \\
&\geq C_1 + C_2 \int_a^L \frac{ds}{r(s)^2} \text{ for some } a \in [0, L) \\
&\geq C_1 + \frac{C_2}{A} \int_a^L \frac{|r'(s)|}{r(s)^2} ds \\
&= C_1 + \frac{C_2}{A} \int_a^0 \frac{-dr}{r^2} \, (b = r(a), \text{ and } r \text{ is decreasing }) = \infty.
\end{aligned}$$

The total curvature τ_f of M is -8π. This is a consequence of the fact that φ covers CP^1 two times. Thus we have shown that the Jorge-Meeks surface

$$CP^1 \backslash \{1, e^{2\pi i/3}, e^{4\pi i/3}\} \to R^3$$

is a complete conformal minimal immersion of finite total curvature -8π. We further observe that the ends are embedded. This is due to the Jorge-Meeks equality which states the $\tau_f = 2\pi(\chi(M) - r)$.

4. Linear Fractional Transformations and the Immersion Theorem

We consider a linear fractional transformation (or a Möbius transformation)

$$S : CP^1 \to CP^1,$$

where $CP^1 = C \cup \{\infty\}$. Such a map is given by

$$S(z) = \frac{az + b}{cz + d},$$

where a, b, c, d are complex numbers with $ad - bc \neq 0$.

We review some properties of Möbius transformations. If S is a Möbius transformation, then S is the composition of translations, dilations, and the inversion. Let z_2, z_3, z_4 be any three points in CP^1. Define $S : CP^1 \to CP^1$ by

$$S(z) = \left(\frac{z - z_3}{z - z_4}\right) / \left(\frac{z_2 - z_3}{z_2 - z_4}\right) \quad \text{if } z_2, z_3, z_4 \in C;$$
$$S(z) = \frac{z - z_3}{z - z_4} \text{ if } z_2 = \infty;$$
$$S(z) = \frac{z_2 - z_4}{z - z_4} \text{ if } z_3 = \infty;$$
$$S(z) = \frac{z - z_3}{z_2 - z_3} \text{ if } z_4 = \infty.$$

In all cases

$$S(z_2) = 1, \quad S(z_3) = 0, \quad S(z_4) = \infty,$$

and S is the only Möbius transformation having this property.

It can be shown that given any pair of three distinct points $\{q_1, q_2, q_3\}$, $\{p_1, p_2, p_3\}$ of CP^1 there exists a (unique) Möbius transformation taking q_i to p_i, $1 \le i \le 3$.

PROPOSITION 4. *Suppose $f : M \to R^3$ is a complete conformal minimal immersion of finite total curvature. Also suppose $\Psi : \widetilde{M} \to M$ is a biholomorphism, where $\widetilde{M}$ is another Riemann surface. The $f \circ \Psi$ is a complete minimal conformal immersion of finite total curvature.*

Proof. Since f is an immersion and Ψ is an immersion, $f \circ \Psi$ is also an immersion. We make an observation: Suppose $F : N_1 \to (N_2, ds^2)$ is a diffeomorphism, where N_1 is a manifold, and ds^2 is a complete metric on N_2. Then $F^* ds^2$ is a complete metric.
Thus

$$(f \circ \Psi)^* ds_E^2 = \Psi^*(f^* ds_E^2) = \Psi^*(ds^2)$$

is a complete metric on $\widetilde{M}$ by the observation.

Let $z = (x, y)$ be isothermal coordinates on M. This means that

$$ds^2 = h(z) dz \cdot d\bar{z}$$

for some local function $h > 0$. Put $w = z \circ \Psi$. Then

$$\begin{aligned}\Psi^* ds^2 &= \Psi^*(h(z)dz \cdot d\bar{z}) \\ &= h(z \circ \Psi)d(z \circ \Psi) \cdot d(\overline{z \circ \Psi}) \\ &= h(w)dw \cdot d\bar{w}.\end{aligned}$$

Since Ψ is biholomorphic, w is a holomorphic coordinate on $\widetilde{M}$. And so $f \circ \Psi$ is conformal.

Let us now look at the Gauss map of $f \circ \Psi$, namely

$$\Phi_{f \circ \Psi} : \widetilde{M} \to CP^2.$$

It is not hard to see that $\Phi_{f \circ \Psi} = \Phi_f \circ \Psi$. Since Φ_f and Ψ are both holomorphic, $\Phi_f \circ \Psi$ is also holomorphic. By proposition 2 then, $f \circ \Psi$ is minimal.

Finally, we will show that $f \circ \Psi$ has finite total curvature. By the Chern-Osserman Theorem, f has finite total curvature if and only if $\Phi_f : M \to CP^2$ is algebraic. Since Φ_f is algebraic, M is biholomorphic to $M_g \backslash \{p_1, \cdots, p_d\}$, where M_g is compact Riemann surface of genus g. Since Ψ is biholomorphic with $\Psi(q_i) = p_i$, $\widetilde{M}$ is biholomorphic to

$$\widetilde{M_g} \backslash \{\Psi^{-1}(p_1), \cdots, \Psi^{-1}(p_d)\} = \widetilde{M_g} \backslash \{q_1, \cdots, q_d\},$$

where $\widetilde{M_g}$ is compact Riemann surface of genus g. Let

$$\Delta_j = \{z \in C : |z| < 1\}$$

be a holomorphic coordinate system for M_g centered at p_j. In $\Delta_j \backslash \{0\}$, the gauss map of f is given by

$$\Phi_f(z) = [\partial f^1/\partial z, \partial f^2/\partial z, \partial f^3/\partial z].$$

Let $\widetilde{\Delta}_j = \{\widetilde{z} \in C : |\widetilde{z}| < 1\}$ be a local holomorphic coordinate system for $\widetilde{M_g}$ centered at q_j. In $\widetilde{\Delta}_j \backslash \{0\}$ we have

$$\Phi_{f \circ \Psi}(\widetilde{z}) = [\frac{\partial (f \circ \Psi)^1}{\partial \widetilde{z}}, \frac{\partial (f \circ \Psi)^2}{\partial \widetilde{z}}, \frac{\partial (f \circ \Psi)^3}{\partial \widetilde{z}}].$$

We want to show that $\Phi_{f\circ\Psi}$ extends to all of $\widetilde{M}_g$. we will show that $\frac{\partial(f\circ\Psi)^i}{\partial\bar{z}}, 1 \leq i \leq 3$, have at most a pole at 0. Observe that if $h : M \to CP^1 = C \cup \{\infty\}$ is a holomorphic map and if $\Psi : \widetilde{M} \to M$ is a biholomorphism, then $h \circ \Psi$ is a holomorphic map; a meromorphic function is just a holomorphic map into CP^1. Consequently, the functions $\frac{\partial(f\circ\Psi)^i}{\partial\bar{z}}$ have at most a pole at 0: if one of the these functions had a pole at 0, then one simply replaces $(\frac{\partial(f\circ\Psi)^i}{\partial\bar{z}})$ by $(\bar{z}^b\frac{\partial(f\circ\Psi)^i}{\partial\bar{z}})$, where b is any integer larger than the maximum order of the pole, thereby "removing" the pole. From this we see that $\Phi_{f\circ\Psi}$ extends to all $\widetilde{M}_g$. So $\Phi_{f\circ\Psi}$ is algebraic and by the Chern-Osserman Theorem, $\tau_{f\circ\Psi} < \infty$.

Combining Proposition 4 with the Jorge-Meeks construction we obtain the

Immersion Theorem. Given any r points $\sum_r$ on CP^1 with $1 \leq r \leq 3$, there exists a complete conformal minimal immersion of finite total curvature (with embedded ends)

$$f : CP^1\backslash\sum_r \to R^3.$$

EXAMPLE. Suppose $\Psi : CP^1\backslash\{1,0,\infty\} \to CP^1\backslash\{1, e^{2\pi i/3}, e^{4\pi i/3}\}$ is a biholomorphism. Let

$$f_0 : CP^1\backslash\{1, e^{2\pi i/3}, e^{4\pi i/3}\} \to R^3$$

denote the Jorge-Meeks surface. Then

$$f_0 \circ \Psi : CP^1\backslash\{1,0,\infty\} \to R^3$$

is a complete conformal minimal immersion with total curvature -8π, and the ends of $CP^1\backslash\{1,0,\infty\}$ are embedded. Explicitly, Ψ is given by

$$\Psi(z) = (a - z)/(az - 1),\ a = (1 - \sqrt{3}i)/2.$$

Let $\{\widetilde{\mu}, \widetilde{\varphi}\}$ be the Weierstrass representative of $\Psi^* f_0 = f_0 \circ \Psi$. Then $\{\widetilde{\mu}, \widetilde{\varphi}\}$ is given by

$$\widetilde{\varphi}(z) = \frac{(a-z)^2}{(az-1)^2},\ \widetilde{\mu}(z) = \frac{(az-1)^4 \cdot (1-a^2)dz}{((a-z)^3 - (az-1)^3)^2}.$$

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