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ON THE GAUSS MAP OF HYPERSURFACES IN $\mathbf{R}^{n+1}$ AND IN $\mathbf{R}_1^{n+1}$

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1. Introduction

Let $x : M^n \to \mathbf{R}^m$ be an isometric immersion of a manifold M^n into the Euclidean space $\mathbf{R}^m$ and Δ its Laplacian defined by $-$ div $\circ$ grad. The family of such immersions satisfying the condition $\Delta x = \lambda x, \lambda \in \mathbf{R}$, is characterized by a well known result of Takahashi, [9]: they are either minimal in $\mathbf{R}^m$ or minimal in some Euclidean hypersphere. Since $\Delta x = -nH$, these submanifolds satisfy the condition $\Delta H = \lambda H$, where H is the mean curvature vector field in $\mathbf{R}^m$. When $m = n+1$, that is, M^n is a hypersurface of $\mathbf{R}^{n+1}$, the family C_λ of such hypersurfaces was explored for some special cases in [2].

In this paper we explore the hypersurfaces in $\mathbf{R}^{n+1}$ and in $\mathbf{R}_1^{n+1}$ satisfying the condition $\Delta G = \lambda G$, where G is the Gauss map of M^n in $\mathbf{R}^{n+1}$ and in $\mathbf{R}_1^{n+1}$. This condition is equivalent to the condition that the coordinate functions $G_1, \ldots, G_{n+1} : M^n \to \mathbf{R}$ are eigenfunctions of the Laplacian Δ of M^n with the same eigenvalue.

The following hypersurfaces in $\mathbf{R}^{n+1}$ satisfy the condition $\Delta G = \lambda G$: hyperplane $\mathbf{R}^n$, sphere $S^n(r)$, cylinder over a round sphere $\mathbf{R}^{n-p} \times S^p(r)$.

In section 2 we classify the hypersurfaces satisfying the condition $\Delta G = \lambda G$ for the following special cases: 1) M is compact, 2) $n = 2$ and λ is arbitrary, 3) n is arbitrary and $\lambda = 0$, 4) $n \geq 4$ and M^n is conformally flat, 5) M has non-negative sectional curvatures.

Let $x : M^n \to \mathbf{R}_\nu^m$ be an immersed semi-Riemannian submanifold of $\mathbf{R}_\nu^m$. Then Markvorsen proved the extended version of Takahashi's theorem, [5], as follows: $\Delta x = \lambda x$ for some constant λ if and only if M^n

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is either minimal in $\mathbf{R}^m_\nu$ or minimal in a connected component of the generalized distance sphere

$$\mathbf{S}^{m-1}_\nu(r) = \{x \in \mathbf{R}^m_\nu : |<x,x>| = r^2\}.$$

In section 3, we classify the spacelike hypersurfaces in $\mathbf{R}^{n+1}_1$ satisfying the condition $\Delta G = \lambda G$ for the following special cases: 1) $n = 2$ and λ is arbitrary, 2) n is arbitrary and $\lambda = 0$.

2. Hypersurfaces in the Euclidean space $\mathbf{R}^{n+1}$

Let M^n be a hypersurface in the Euclidean space $\mathbf{R}^{n+1}$. We denote by $\sigma, A, H, \nabla, \overline{\nabla}$ and G the second fundamental form, the Weingarten map, the mean curvature vector field $\frac{1}{n}tr(\sigma)$, the Riemannian connection of M^n, the flat connection of $\mathbf{R}^{n+1}$ and a locally defined unit normal vector field of M^n in $\mathbf{R}^{n+1}$. Then we have the following. For the proof see the lemma 3.2 in [1].

LEMMA 2.1. $\Delta G = n\nabla\alpha + |A|^2 G$, *where* α *is the mean curvature with respect to* G, Δ *is the Laplacian of* M^n *acting on* $(n+1)$*-valued functions and* $|A|^2 = tr(A^2)$.

REMARK 2.2. From the above lemma, we may obtain the following formula in [2],

$$\Delta H = (\Delta\alpha + \alpha|A|^2)G + n\alpha\nabla\alpha + 2A(\nabla\alpha),$$

as follows. Let $G = (G_1, \ldots, G_{n+1}) = \sum_{i=1}^{n+1} G_i e_i$. Then we have $\Delta H = \Delta(\alpha G) = \sum_{i=1}^{n+1} \Delta(\alpha G_i) e_i$. Since $\Delta(\alpha G_i) = (\Delta\alpha)G_i + \alpha\Delta G_i - 2 <\nabla\alpha, \nabla G_i>$,

$$\Delta H = (\Delta\alpha)G + \alpha\Delta G - 2\sum_{i=1}^{n+1} <\nabla\alpha, \nabla G_i> e_i.$$

On the other hand, choose local orthonormal eigenvectors $X_1, \ldots, X_n$ of

A with eigenvalues $\mu_1, \ldots, \mu_n$, respectively. Then, we have

$$\sum_{i=1}^{n+1} \langle \nabla\alpha, \nabla G_i \rangle e_i = \sum_{i=1}^{n+1} \sum_{j=1}^{n} X_j(\alpha) X_j(G_i) e_i = \sum_{j=1}^{n} X_j(\alpha) X_j(G)$$
$$= \sum_{j=1}^{n} X_j(\alpha) \overline{\nabla}_{X_j}(G) = -\sum_{j=1}^{n} X_j(\alpha) A(X_j)$$
$$= -\sum_{j=1}^{n} \mu_j X_j(\alpha) X_j = -A(\nabla\alpha).$$

And since $\Delta G = n\nabla\alpha + |A|^2 G$, the formula follows.

Using the above lemma, since $\nabla\alpha$ is tangent to M and G is normal to M, we have the following. For the proof see the theorem 4.1 and the theorem 4.2 in [1].

PROPOSITION 2.3. *Let M^n be a hypersurface in $\mathbf{R}^{n+1}$. Then M^n has constant mean curvature α if and only if $\Delta G = |A|^2 G$. And $\Delta G = \lambda G$ for some constant λ if and only if the mean curvature α and $|A|^2$ are constant.*

PROPOSITION 2.4. *Let $x : M^n \to \mathbf{R}^{n+1}$ be an immersed compact hypersurface in $\mathbf{R}^{n+1}$. Then $\Delta G = \lambda G$ if and only if M^n is a sphere $S^n(r)$.*

For the case $n = 2$, suppose $\Delta G = \lambda G$. Then $2\alpha = \mu_1 + \mu_2$ and $\lambda = \mu_1^2 + \mu_2^2$ are constant. Hence A has constant eigenvalues. That is, M^2 is an isoparametric hypersurface in $\mathbf{R}^3$. From the fact that the isoparametric surfaces in $\mathbf{R}^3$ are $\mathbf{R}^2$, $S^2(r)$, $S^1(r) \times \mathbf{R}$,we obtain the following.

THEOREM 2.5. *Let M^2 be a surface of $\mathbf{R}^3$. Then $\Delta G = \lambda G$ for some constant λ if and only if M^2 is an open portion of one of the following surfaces:*

$$R^2 \quad (\lambda = 0), \quad S^2(r) \quad (\lambda = \frac{2}{r^2}), \quad S^1(r) \times R \quad (\lambda = \frac{1}{r^2}).$$

If $\Delta G = 0$, then $\lambda = |A|^2 = \mu_1^2 + \cdots + \mu_n^2 = 0$. Hence M^n is totally geodesic and we obtain the following.

THEOREM 2.6. *The only hypersurfaces of* $\mathbf{R}^{n+1}$ *satisfying* $\Delta G = 0$ *are the hyperplanes.*

If M^n is a conformally flat hypersurface of R^{n+1} which is not totally umbilic and $n \geq 4$, then, by the Theorem 3 of [6] the Weingarten map A has two distinct eigenvalues μ_1, μ_2 of multiplicities 1 and $n-1$, respectively. Suppose that $\Delta G = \lambda G$. Then $n\alpha = \mu_1 + (n-1)\mu_2$ and $\lambda = \mu_1^2 + (n-1)\mu_2^2$ are constant. Hence A has constant eigenvalues. That is, M^n is an isoparametric hypersurface in $\mathbf{R}^{n+1}$. From the fact that the isoparametric hypersurfaces in $\mathbf{R}^{n+1}$ are R^n, $S^n(r)$ and $S^p(r) \times \mathbf{R}^{n-p}$ ([7]), we obtain the following.

THEOREM 2.7. *Let* M^n *be a conformally flat hypersurface in* $\mathbf{R}^{n+1}$, $n \geq 4$. *Then* M^n *satisfies* $\Delta G = \lambda G$ *for some constant* λ *if and only if* M^n *is an open portion of one of the following hypersurfaces:*

(1) *In case* M^n *is totally umbilic,*

$$\mathbf{R}^n \quad (\lambda = 0), \qquad S^n(r) \quad (\lambda = \frac{n}{r^2})$$

(2) *In case* M^n *is not totally umbilic,*

$$\mathbf{R} \times S^{n-1}(r) \quad (\lambda = \frac{n-1}{r^2}), \qquad \mathbf{R}^{n-1} \times S^1(r) \quad (\lambda = \frac{1}{r^2}).$$

THEOREM 2.8. *Let* M^n *be a hypersurface of* $\mathbf{R}^{n+1}$ *with nonnegative sectional curvatures. Then* $\Delta G = \lambda G$ *for some constant* λ *if and only if* M^n *is an open portion of one of the following hypersurfaces:*

$$\mathbf{R}^n \quad (\lambda = 0), \qquad S^n(r) \quad (\lambda = \frac{n}{r^2}), \qquad S^p(r) \times \mathbf{R}^{n-p} \quad (\lambda = \frac{p}{r^2}).$$

Proof. Suppose that $\Delta G = \lambda G$ for some constant λ. Then the mean curvature α and the square $|A|^2$ of the length of the second fundamental form are constant. Hence by the formula (18) in [8], we have

$$\sum_{i<j} (\mu_i - \mu_j)^2 K_{ij} + |\nabla A|^2 = 0,$$

where $\mu_1, \ldots, \mu_n$ is the eigenvalues of A in the direction $X_1, \ldots, X_n$ and K_{ij} is the sectional curvature of the section spanned by X_i, X_j. Since $K_{ij} \geq 0$, the second fundamental form is parallel. Thus the principal curvatures of M are constant, that is, M is an isoparametric hypersurface of $\mathbf{R}^{n+1}$. Hence M is an open portion of one of the above hypersurfaces. The converse is obvious.

3. Spacelike hypersurfaces in the Minkowski space $\mathbf{R}_1^{n+1}$

Now, if the ambient space is the Minkowski space $\mathbf{R}_1^{n+1}$ with metric $ds^2 = -dx_1^2 + dx_2^2 + \cdots + dx_{n+1}^2$ and if the hypersurface M^n is spacelike,then there is a globally defined future-directed unit timelike vector field G on M^n. And we may consider the condition $\Delta G = \lambda G$ on M. First, we prove the following lemma analogous to Lemma 2.1.

LEMMA 3.1. *$\Delta G = -n\nabla\alpha - |A|^2 G$, where α is the mean curvature with respect to G. Δ is the Laplacian of M^n acting on $(n+1)$-valued functions and $|A|^2 = tr(A^2)$.*

Proof. As in the proof of Lemma 2.1, choose an orthonormal local frame $X_1, \ldots, X_{n+1}$ in such a way that $X_1, \ldots, X_n$ are tangent to M and X_{n+1} is the unit timelike vector field G. Moreover we may assume that $X_1, \ldots, X_n$ are eigenvectors of $A = A_G$ with the eigenvalues μ_i, $AX_i = \mu_i X_i$, $i = 1, \ldots, n$. Denote by $w^1, \ldots, w^{n+1}$ and $w_i^j, i, j = 1, \ldots, n+1$, the dual frame and connection forms associated to $X_1, \ldots, X_{n+1}$, respectively. Note that $w_i^j = -w_j^i$, $w_i^{n+1} = w_{n+1}^i$, for $i, j = 1, \ldots, n$ and $w_{n+1}^{n+1} = 0$. Then, using the connection equations we obtain

$$-\mu_j X_j = -A(X_j) = \overline{\nabla}_{X_j} X_{n+1} = \sum_{k=1}^{n} \omega_{n+1}^k(X_j) X_k.$$

Hence we have

$$w_k^{n+1}(X_j) = w_{n+1}^k(X_j) = -\delta_{jk}\mu_j.$$

Now, as before,

$$\begin{aligned}\Delta G &= \sum_{j=1}^{n}\{\overline{\nabla}_{\nabla_{X_j}X_j}X_{n+1} - \overline{\nabla}_{X_j}\overline{\nabla}_{X_j}X_{n+1}\}\\ &= \sum_{j=1}^{n}\{-\sum_{k=1}^{n}\mu_k w_j^k(X_j)X_k + X_j(\mu_j)X_j + \mu_j\overline{\nabla}_{X_j}X_j\}\\ &= \sum_{k=1}^{n}\{X_k(\mu_k) + \sum_{j=1}^{n}(\mu_j - \mu_k)w_j^k(X_j)\}X_k - \sum_{k=1}^{n}\mu_k^2 X_{n+1},\end{aligned}$$

where we use the equation

$$\overline{\nabla}_{X_j}X_j = \sum_{k=1}^{n} w_j^k(X_j)X_k + w_j^{n+1}(X_j)X_{n+1} = \sum_{k=1}^{n} w_j^k(X_j)X_k - \mu_j X_{n+1}.$$

But, as in the Euclidean case, by the Codazzi's equations we have

$$X_k(\mu_j) = (\mu_j - \mu_k)w_j^k(X_j) \quad \text{for distinct} \quad j,k = 1,\dots,n.$$

Consequently

$$\Delta G = \sum_{k=1}^{n} X_k(\sum_{j=1}^{n}\mu_j)X_k - \sum_{k=1}^{n}\mu_k^2 X_{n+1},$$

and since $n\alpha = -tr(A) = -\sum_{j=1}^{n}\mu_j$, $|A|^2 = \sum_{k=1}^{n}\mu_k^2$, the lemma follows.

REMARK 3.2. From the above formula, as before, we may obtain the following formula. See the formula (1.8) in [3].

$$H = \alpha G, \quad \Delta H = (\Delta\alpha - \alpha|A|^2)G - n\alpha\nabla\alpha + 2A(\nabla\alpha).$$

From the above lemma, we have the following.

THEOREM 3.3. *Let M^n be a spacelike hypersurface in the Minkowski space $\mathbf{R}_1^{n+1}$. Then $\Delta G = \lambda G$ for some constant λ if and only if the mean curvature α and $|A|^2 = tr(A^2)$ are constant.*

THEOREM 3.4. *Let M^2 be a spacelike surface of $\mathbf{R}_1^3$. Then $\Delta G = \lambda G$ for some constant λ if and only if M^2 is an open portion of one of the following surfaces:*

$$R^2\ (\lambda = 0), \quad H^2(r)\ (\lambda = -\frac{2}{r^2}), \quad H^1(r) \times R\ (\lambda = -\frac{1}{r^2}).$$

Proof. Suppose that $\Delta G = \lambda G$. Then $2\alpha = \mu_1 + \mu_2$ and $-\lambda = |A|^2 = \mu_1^2 + \mu_2^2$ are constant, where μ_1, μ_2 are the eigenvalues of the shape operator A. Hence the eigenvalues μ_1, μ_2 of A are constant and M^2 is an open portion of one of the above surfaces (see [4], p. 175).

Conversely, for $R^2 = \{(x_1, x_2, x_3) \in \mathbf{R}_1^3 : x_1 = 0\}$, $G = (1, 0, 0)$ is constant. Hence $\Delta G = 0$. Note that

$$H^2(r) = \{(x_1, x_2, x_3) \in \mathbf{R}_1^3 : -x_1^2 + x_2^2 + x_3^2 = -r^2, x_1 > 0\},$$

$$H^1(r) \times R = \{(x_1, x_2, x_3) \in \mathbf{R}_1^3 : -x_1^2 + x_2^2 = -r^2, x_1 > 0\}.$$

Let $f(x_1, x_2, x_3) = -x_1^2 + x_2^2 + \delta x_3^2, \delta = 0, 1$, then the above surfaces are $f^{-1}(-r^2)$. A straight computation shows that $G = (1/r)(x_1, x_2, \delta x_3)$, $\mu_1 = -1/r$, $\mu_2 = -\delta/r$. Then, the mean curvature is given by

$$\alpha = -\frac{(\mu_1 + \mu_2)}{2} = \frac{1}{2r}(1 + \delta)$$

and

$$|A|^2 = \mu_1^2 + \mu_2^2 = \frac{1}{r^2}(1 + \delta).$$

Therefore, by using Lemma 3.1, we have $\Delta G = 1/r^2(1 + \delta)G$ and $\lambda = -(1 + \delta)/r^2$.

If $\Delta G = 0$, then $\mu_1^2 + \cdots + \mu_n^2 = |A|^2 = -\lambda = 0$. Hence M^n is totally geodesic and we obtain the following.

THEOREM 3.5. *The only spacelike hypersurfaces in the Minkowski space $\mathbf{R}_1^{n+1}$ satisfying $\Delta G = 0$ are the hyperplanes.*

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