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ON A NEARLY KAEHLERIAN FINSLER STRUCTURE

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1. Introduction

Let M be an n-dimensional Finsler manifold with the fundamental function $L(x,y)$ homogeneous of the first degree in y. The metric tensor $g_{ij}(x,y)$ of M is introduced by $g_{ij}(x,y) = 1/2\,\dot{\partial}_i\dot{\partial}_j L^2(x,y)$, where $\dot{\partial}_i = \partial/\partial y^i$. We assume that M admits an almost complex structure $f^i{}_j(x)$ which depend on a point x of M and $L(x,y)$ satisfies Rizz's condition

$$(1.1)\qquad L(x, y\cos\theta + f(x)y\sin\theta) = L(x,y)$$

for any θ. If we put $\phi^i_{\theta j} = \cos\theta\delta^i{}_j + \sin\theta f^i{}_j(x)$, then the condition (1.1) is expressed as

$$(1.2)\qquad L(x, \phi_\theta y) = L(x,y).$$

Since $L(x,ky) = kL(x,y)$ for any positive k, (1.2) is rewritten as

$$(1.3)\qquad L(x,cy) = |c|L(x,y)$$

for any non-zero complex number c.

The manifold which admits a Finsler metric $g_{ij}(x,y)$ and an almost complex structure $f^i{}_j(x)$ satisfying the condition (1.2) or (1.3) is called an almost Hermitian Finsler manifold or simply a Rizza manifold and the structure $(f^i{}_j(x), g_{ij}(x,y))$ is called an almost Hermitian Finsler structure or a Rizza structure.

With respect to the Rizza condition, Ichijyō [3] has shown that the Rizza condition (1.2) is equivalent to

$$(1.4)\qquad g_{im}(x,y)f^m{}_j(x) + g_{jm}(x,y)f^m{}_i(x) + 2C_{ijm}f^m{}_r(x)y^r = 0,$$

where $C_{ijm}(x,y) = 1/2\,\dot{\partial}_m g_{ij}(x,y)$. And Fukui [1] has proved

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THEOREM A. *If a Finsler metric $g_{ij}(x,y)$ and almost complex structure $f^i{}_j(x)$ satisfy the condition*

$$g_{pq}(x,y)f^p{}_i(x)f^q{}_j(x) = g_{ij}(x,y), \tag{1.5}$$

then g_{ij} is a Riemannian metric, that is, (f,g) is an almost Hermitian structure.

In a Rizza manifold, we put $f_{ij}(x,y) = g_{im}(x,y)f^m{}_j(x)$. If $f_{ij}(x,y) + f_{ji}(x,y) = 0$, then we obtain the condition (1.5) easily, that is, the Rizza manifold is an almost Hermitian manifold.

Let $\overset{*}{\nabla}_k$ be the h-covariant derivative for the Cartan's Finsler connection. A Rizza manifold satisfying $\overset{*}{\nabla}_k f^i{}_j = 0$ is said to be a Kaehlerian Finsler manifold. With respects to the Kaehlerian Finsler manifold, Ichjyō [3] and Fukui [1] have studied.

In the present paper, we are concern with a Rizza manifold satisfying the condition $\overset{*}{\nabla}_k f^i{}_j + \overset{*}{\nabla}_j f^i{}_k = 0$ for the Cartan's Finsler connection following the example of complex Riemannian geometry. And some properties of a nearly Kaehlerian $(f,\tilde{g},N)$-structure and a quasi nearly Kaehlerian Finsler manifold (defined in Section 3) are investigated.

2. A nearly Kaehlerian Finsler manifold

Let M be a Rizza manifold with a Rizza structure $(f^i{}_j(x), g_{ij}(x,y))$ and $\overset{*}{\nabla}_k$ be a h-covariant derivative for the Cartan's Finsler connection $(\overset{*}{\Gamma}{}^i{}_{jk}, G^i{}_j, C^i{}_{jk})$. For an any Finsler tensor $T^i{}_j$, we have

$$\overset{*}{\nabla}_k T^i{}_j = \partial_k T^i{}_j - G^m{}_k \dot{\partial}_m T^i{}_j + \overset{*}{\Gamma}{}^i{}_{mk} T^m{}_j - T^i{}_m \overset{*}{\Gamma}{}^m{}_{jk},$$

where $\partial_k = \partial/\partial x^k$. Therefore, with respect to the almost complex structure tensor $f^i{}_j(x)$, we have

$$\overset{*}{\nabla}_k f^i{}_j = \partial_k f^i{}_j + \overset{*}{\Gamma}{}^i{}_{mk} f^m{}_j - f^i{}_m \overset{*}{\Gamma}{}^m{}_{jk}. \tag{2.1}$$

Now a Rizza manifold satisfying the condition

$$\overset{*}{\nabla}_k f^i{}_j + \overset{*}{\nabla}_j f^i{}_k = 0 \tag{2.2}$$

is said to be a nearly Kaehlerian Finsler manifold and the Rizza structure $(f^i{}_j(x), g_{ij}(x,y))$ satisfying (2.2) is called a nearly Kaehlerian Finsler structure.

We put

$$\Omega = f_{ij}(x,y)dx^i \wedge dx^j. \tag{2.3}$$

This 2-form is globally defined on tangent bundle $T(M)$.

Since $d\Omega = \partial_k f_{ij} dx^k \wedge dx^i \wedge dx^j + \dot{\partial}_k f_{ij} dy^k \wedge dx^i \wedge dx^j$, the condition $d\Omega = 0$ can be written as

$$\text{(2.4)} \quad \begin{aligned} &(1) \quad \partial_k f_{ij} + \partial_i f_{jk} + \partial_j f_{ki} - \partial_k f_{ji} - \partial_i f_{kj} - \partial_j f_{ik} = 0, \\ &(2) \quad \dot{\partial}_k f_{ij} - \dot{\partial}_k f_{ji} = 0. \end{aligned}$$

The condition (2) of (2.4) implies $C_{kim} f^m{}_j = C_{kjm} f^m$, from which $C_{kjm} f^m{}_r y^r = 0$ by virtue of $C_{kjm} y^j = 0$. So, from (1.4) we have $f_{ij}(x,y) + f_{ji}(x,y) = 0$. In accordance with Theorem A, it follows that g_{ij} is a Riemannian metric, that is, (f,g) defines an almost Hermitian structure. Then we can find $\overset{*}{\Gamma}{}^i{}_{jk} = \left\{ \begin{matrix} i \\ jk \end{matrix} \right\}$. In this case, the condition (1) of (2.4) is induced to

$$\nabla_k f_{ij} + \nabla_i f_{jk} + \nabla_j f_{ki} = 0 \tag{2.5}$$

by virtue of $\overset{*}{\nabla}_k f_{ij} = \nabla_k f_{ij}$ and $f_{ij} = -f_{ji}$, where ∇_k is the covariant derivative for the Levi-Civita connection. From conditions (2.2) and (2.5) we have $\nabla_k f^i{}_j = 0$. Conversely, if (f,g) is a Kaehlerian structure, then the condition (1) and (2) of (2.4) are satisfied evidently.

Thus we have

THEOREM 2.1. *Let M be a nearly Kaehlerian Finsler manifold whose nearly Kaehlerian Finsler structure is given by (f,g), and Ω be 2-form defined by (2.3). In order that $d\Omega = 0$ holds, it is necessary and sufficient that (f,g) is a Kaehlerian structure.*

As well as the Cartan's Finsler connection $(\overset{*}{\Gamma}{}^i{}_{jk}, G^i{}_j, C^i{}_{jk})$, we know the Berwald's Finsler connection $(G^i{}_{jk}, G^i{}_j, 0)$, where $G^i(x,y)$ is (2) p-homogeneous function in y induced from the geodesic equation on a Finsler manifold, $G^i{}_j = \dot{\partial}_j G^i$ and $G^i{}_{jk} = \dot{\partial}_k G^i{}_j$.

We put $G^h{}_{ijk} = \dot{\partial}_i G^h{}_{jk}$ and $G_{ij} = G^r{}_{ijr}$. It is noted that $G^h{}_{ijk}$ and G_{ij} are symmetric in indices i, j, k and i, j respectively.

By Euler's theorem on homogeneous function in y, we have

$$(2.6)\qquad \begin{aligned} &(1)\quad G^h{}_{ij0} = G^h{}_{i0j} = G^h{}_{0ij} = 0,\\ &(2)\quad G^h{}_{i0} = G^h{}_{0i} = G^h{}_i,\\ &(3)\quad G_{0j} = G_{j0} = 0,\\ &(4)\quad G^h{}_0 = 2G^h, \end{aligned}$$

where the index 0 denote the contraction with the element of support y. From (3) of (2.6), we obtain

$$(2.7)\qquad (\dot{\partial}_r G_{jm}) y^m = -G_{jr}.$$

If $G^h{}_{ij}$ are functions of position alone, namely $G^h{}_{ijk} = 0$ holds, then the Finsler manifold is said to be a Berwald space. The tensor field D with the components

$$D^h{}_{ijk} = G^h{}_{ijk} - 1/(n+1)(y^h \dot{\partial}_k G_{ij} + \delta^h{}_i G_{jk} + \delta^h{}_j G_{ki} + \delta^h{}_k G_{ij})$$

is known as Douglas tensor [4]. This tensor is invariant under the projective change of a Finsler manifold.

THEOREM 2.2. *In a nearly Kaehlerian Finsler manifold M with vanishing Douglas tensor, if $G^i{}_r f^m{}_\ell$ are symmetric in r and ℓ, then M is a Berwald space.*

Proof. From the assumption we have

$$(2.8)\qquad \partial_k f^i{}_j + \partial_j f^i{}_k + \overset{*}{\Gamma}{}^i{}_{mk} f^m{}_j + \overset{*}{\Gamma}{}^i{}_{mj} f^m{}_k - 2 f^i{}_m \overset{*}{\Gamma}{}^m{}_{jk} = 0,$$

$$(2.9)\qquad G^h{}_{ijk} = 1/(n+1)(y^h \dot{\partial}_k G_{ij} + \delta^h{}_i G_{jk} + \delta^h{}_j G_{ki} + \delta^h{}_k G_{ij}).$$

Transvecting (2.8) with y^j and y^k successively, we have

$$(2.10)\qquad y^j y^k \partial_j f^i{}_k + G^i{}_m f^m{}_0 - 2G^m f^i{}_m = 0$$

by virtue of (2.6) and $\overset{*}{\Gamma}{}^i{}_{0j} = G^i{}_j$.

Differentiating (2.10) partially with respect to y^r and y^ℓ successively, we have

$$(2.11)\quad \partial_r f^i{}_\ell + \partial_\ell f^i{}_r + G^i{}_{\ell mr} f^m{}_0 + G^i{}_{mr} f^m{}_\ell + G^i{}_{m\ell} f^m{}_r - 2G^m{}_{r\ell} f^i{}_m = 0.$$

If $G^i{}_r f^m{}_\ell$ are symmetric in r and ℓ, then we have $G^i{}_{rh} f^m{}_\ell = G^i{}_{\ell h} f^m{}_r$, from which

$$(2.12)\qquad G^i{}_{rm} f^m{}_\ell = G^i{}_{\ell m} f^m{}_r.$$

Differentiating (2.12) with respect to y^k, we get

$$(2.13)\qquad G^i{}_{rmk} f^m{}_\ell = G^i{}_{\ell mk} f^m{}_r,$$

from which

$$(2.14)\qquad G_{rm} f^m{}_\ell = G_{\ell m} f^m{}_r.$$

Transvecting (2.13) and (2.14) with y^ℓ respectively and using (2.6), we have

$$(2.15)\qquad \begin{aligned} &(1)\quad G^i{}_{rmh} f^m{}_0 = 0, \\ &(2)\quad G_{rm} f^m{}_0 = 0. \end{aligned}$$

Differentiating (2) of (2.15) with respect to y^h, we obtain

$$(2.16)\qquad (\dot\partial_m G_{rh}) f^m{}_0 = -G_{rm} f^m{}_h.$$

Substituting (2.12) and (1) of (2.15) into (2.11), we have

$$(2.17)\qquad \partial_r f^i{}_\ell + \partial_\ell f^i{}_r + 2G^i{}_{mr} f^m{}_\ell - 2G^m{}_{r\ell} f^i{}_m = 0.$$

Differentiating (2.17) with respect to y^j, we have

$$(2.18)\qquad G^i{}_{mrj} f^m{}_\ell - G^m{}_{r\ell j} f^i{}_m = 0.$$

Therefore, substituting (2.9) into (2.18), we have

$$\begin{aligned} &(y^i \dot\partial_j G_{mr} + \delta^i{}_m G_{rj} + \delta^i{}_r G_{jm} + \delta^i{}_j G_{mr}) f^m{}_\ell \\ &\qquad - (y^m \dot\partial_j G_{r\ell} + \delta^m{}_r G_{\ell j} + \delta^m{}_\ell G_{jr} + \delta^m{}_j G_{r\ell}) f^i{}_m = 0. \end{aligned}$$

Contracting the above equation with respect to i and j, we have $G_{mr} f^m{}_\ell = 0$ by virtue of (2.7), (2.14) and (2.16), that is $G_{ij} = 0$. Thus we obtain $G^h{}_{ijk} = 0$. Consequently M is a Berwald space.

3. A nearly Kaehlerian $(f, \tilde{g}, N)$-structure

The generalized Finsler metric $\tilde{g}_{ij}(x,y)$ satisfying the conditions: $\tilde{g}_{ij}(x,y) = \tilde{g}_{ji}(x,y)$, $\tilde{g}_{ij}(x,y)\xi^i\xi^j$ is positive definite and $\tilde{g}_{ij}(x,y)$ is (0)p-homogeneous for y is called a Moór metric. We can induce the Moór metric $\tilde{g}_{ij}(x,y)$ from a Rizza structure $(f^i{}_j(x), g_{ij}(x,y))$ as

$$\tilde{g}_{ij}(x,y) = 1/2(g_{ij}(x,y) + g_{pq}(x,y)f^p{}_i(x)f^q{}_j(x)) \tag{3.1}$$

and it satisfies

$$\tilde{g}_{pq}(x,y)f^p{}_i(x)f^q{}_j(x) = \tilde{g}_{ij}(x,y). \tag{3.2}$$

Let us assume that an n-dimensional manifold M equipped with a non-linear connection $N^i{}_j(x,y)$. The transformation rule of $N^i{}_j(x,y)$ is given by

$$N^a{}_c(\bar{x},\bar{y})\frac{\partial \bar{x}^c}{\partial x^i} + \frac{\partial^2 \bar{x}^a}{\partial x^i \partial x^m}y^m = \frac{\partial \bar{x}^a}{\partial x^m}N^m{}_i(x,y).$$

The quantity $G^i{}_j$ treated in section 2 is a kind of non-linear connection. Moreover we assume that the manifold M admits an almost complex structure $f^i{}_j(x)$ and a Moór metric $\tilde{g}_{ij}(x,y)$ satisfying (3.2). We will say, in this case, that M admits an $(f, \tilde{g}, N)$-structure. Here the $\tilde{g}_{ij}(x,y)$ is not the induced Moór metric from a Rizza structure.

Let $\tilde{\Gamma}^i{}_{jk} = 1/2\tilde{g}^{im}(X_k\tilde{g}_{jm} + X_j\tilde{g}_{mk} - X_m\tilde{g}_{kj})$ and $\tilde{F}_{ijk} = X_i\tilde{f}_{jk} + X_j\tilde{f}_{ki} + X_k\tilde{f}_{ij}$, where $X_k = \partial_k - N^m{}_k\dot{\partial}_m$ and $\tilde{f}_{ij} = \tilde{g}_{im}f^m{}_j$.

In [3] the following equations are obtained

$$(3.3)\qquad \begin{aligned} &(1)\quad \tilde{\nabla}_k\tilde{g}_{ij} = 0,\\ &(2)\quad \tilde{f}_{ij} = -\tilde{f}_{ji},\\ &(3)\quad \tilde{f}_{im}f^m{}_j = -\tilde{g}_{ij},\\ &(4)\quad \tilde{F}_{ijk} = \tilde{\nabla}_i\tilde{f}_{jk} + \tilde{\nabla}_j\tilde{f}_{ki} + \tilde{\nabla}_k\tilde{f}_{ij}, \end{aligned}$$

where $\tilde{\nabla}_k$ is the h-covariant derivative with respect to $(\tilde{\Gamma}^i{}_{jk}, N^i{}_j)$.

On the other hand, the Nijenhuis tensor of almost complex structure can be expressed, from definition of $\tilde{\nabla}_k$, as

$$(3.4)\qquad N^h{}_{ij} = (\tilde{\nabla}_r f^h{}_i) f^r{}_j - (\tilde{\nabla}_r f^h{}_j) f^r{}_i + f^h{}_r \tilde{\nabla}_i f^r{}_j - f^h{}_r \tilde{\nabla}_j f^r{}_i.$$

This tensor may be written in the form

$$(3.5)\quad \begin{aligned} N^h{}_{ij} = {} & 4 f^h{}_r \tilde{\nabla}_i f^r{}_j - 2 f^h{}_r (\tilde{\nabla}_i f^r{}_j + \tilde{\nabla}_j f^r{}_i) \\ & - (\tilde{\nabla}_j f^h{}_r + \tilde{\nabla}_r f^h{}_j) f^r{}_i + (\tilde{\nabla}_r f^h{}_i + \tilde{\nabla}_i f^h{}_r) f^r{}_j. \end{aligned}$$

Now, an $(f, \tilde{g}, N)$-structure satisfying

$$(3.6)\qquad \tilde{\nabla}_k f^i{}_j + \tilde{\nabla}_j f^i{}_k = 0$$

is said to be a nearly Kaehlerian $(f, \tilde{g}, N)$-structure.

Let us put $\tilde{N}_{hij} = \tilde{g}_{hm} N^m{}_{ij}$. Then, we have

$$(3.7)\qquad \tilde{N}_{hij} = f^r{}_j \tilde{F}_{rih} - f^r{}_i \tilde{F}_{rhj} - 2 \tilde{f}_{jr} \tilde{\nabla}_h f^r{}_i$$

by virtue of (3.3) and (3.4). Therefore

$$(3.8)\qquad \tilde{N}_{hij} + \tilde{N}_{ihj} = -f^r{}_i \tilde{F}_{rhj} - f^r{}_h \tilde{F}_{rij} - 2 \tilde{f}_{jr} (\tilde{\nabla}_h f^r{}_i + \tilde{\nabla}_i f^r{}_h).$$

Hence, if $\tilde{N}_{hij} + \tilde{N}_{ihj} = 0$ and $f^r{}_i \tilde{F}_{rhj} + f^r{}_h \tilde{F}_{rij} = 0$, then (3.6) holds, that is the $(f, \tilde{g}, N)$-structure is a nearly Kaehlerian $(f, \tilde{g}, N)$-structure.

Conversely, if the $(f, \tilde{g}, N)$-structure is a nearly Kaehlerian $(f, \tilde{g}, N)$-structure, then, from (3.5), we have $\tilde{N}_{hij} = 4 \tilde{f}_{hr} \tilde{\nabla}_i f^r{}_j$. Therefore

$$\begin{aligned} \tilde{N}_{hij} + \tilde{N}_{ihj} &= 4(\tilde{f}_{hr} \tilde{\nabla}_i f^r{}_j + \tilde{f}_{ir} \tilde{\nabla}_h f^r{}_j) \\ &= 4 \tilde{f}_{mj} (\tilde{\nabla}_i f^m{}_h + \tilde{\nabla}_h f^m{}_i) = 0, \end{aligned}$$

$$\begin{aligned} f^r{}_i \tilde{F}_{rhj} + f^r{}_h \tilde{F}_{rij} &= f^r{}_i \tilde{\nabla}_h \tilde{f}_{jr} + f^r{}_h \tilde{\nabla}_i \tilde{f}_{jr} \\ &\quad + 2(f^r{}_i \tilde{\nabla}_j \tilde{f}_{rh} + f^r{}_h \tilde{\nabla}_j \tilde{f}_{ri}) \\ &= -\tilde{f}_{jr} (\tilde{\nabla}_h f^r{}_i + \tilde{\nabla}_i f^r{}_h) = 0 \end{aligned}$$

by virtue of (3.3), (3.6) and $f^r{}_i \tilde{\nabla}_j f_{rh} + f^r{}_h \tilde{\nabla}_j f_{ri} = 0$. Consequently we obtain

THEOREM 3.1. *In order that $(f,\tilde{g},N)$-structure is a nearly Kaehlerian $(f,\tilde{g},N)$-structure, it is necessary and sufficient that $\tilde{N}_{ihj}$ and $f^r{}_i\tilde{F}_{rhj}$ are skew-symmetric in i and h respectively.*

Now, let us assume that a manifold M admits a Rizza structure (f,g) and let $\tilde{g}_{ij}(x,y)$ be the induced Moór metric from Rizza structure (f,g). Since the quantity $G^i{}_j$ treated in section 2 is a non-linear connection, $(f^i{}_j, \tilde{g}_{ij}, G^i{}_j)$ determines an $(f,\tilde{g},N)$-structure which we call an $(f,\tilde{g},N)$-structure derived from a Rizza structure.

If the $(f,\tilde{g},N)$-structure is a nearly Kaehlerian $(f,\tilde{g},N)$-structure, then the original Rizza structure (f,g) is said to be a quasi nearly Kaehlerian Finsler structure. From Theorem 3.1, the condition for a Rizza structure (f,g) to be a quasi nearly Kaehlerian Finsler structure is given by $\tilde{\nabla}_k f^i{}_j + \tilde{\nabla}_j f^i{}_k = 0$ and it is equivalent to

$$f^r{}_i\tilde{F}_{rhj} + f^r{}_h\tilde{F}_{rij} = 0, \quad \tilde{N}_{hij} + \tilde{N}_{ihj} = 0. \tag{3.9}$$

On the other hand, the induced Moór metric $\tilde{g}_{ij}(x,y)$ from the Rizza structure may be given (3.1). From it, we have

$$\tilde{f}_{ij} = 1/2(f_{ij} - f_{ji}). \tag{3.10}$$

Since $G^i{}_j = N^i{}_j$ we have

$$\tilde{F}_{ijk} = 1/2(\delta_i f_{jk} + \delta_j f_{ki} + \delta_k f_{ij} - \delta_i f_{kj} - \delta_j f_{ik} - \delta_k f_{ji}),$$

where $\delta_i = \partial_i - G^m{}_i\dot{\partial}_m$.

Putting

$$F_{ijk} = 1/2(\overset{*}{\nabla}_i f_{jk} + \overset{*}{\nabla}_j f_{ki} + \overset{*}{\nabla}_k f_{ij} - \overset{*}{\nabla}_i f_{kj} - \overset{*}{\nabla}_j f_{ik} - \overset{*}{\nabla}_k f_{ji},$$

we can find $\tilde{F}_{ijk} = F_{ijk}$. From (3.1), we have

$$\tilde{N}_{hij} + \tilde{N}_{ihj} = 1/2\left\{N_{hij} + N_{ihj} + f^q{}_m(f_{qi}N^m{}_{hj} + f_{qh}N^m{}_{ij})\right\},$$

where $N_{hij} = g_{hm}N^m{}_{ij}$. Thus we have

THEOREM 3.2. *A Rizza manifold is a quasi nearly Kaehlerian Finsler manifold if and only if $f^r{}_iF_{rhj}$ and $N_{ihj} + f^q{}_m f_{qi}N^m{}_{hj}$ are skew-symmetric in i and h respectively.*

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