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THE WEAKLY TAYLOR INVERTIBILITY FOR A COMMUTING PAIR OF ELEMENTS IN A C^*–ALGEBRA

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Throughout this paper suppose A is a C^*-algebra with unity. By a *chain* of two elements in A we mean a pair $(b, a) \in A^2$ for which $ba = 0$. Whether or not the chain condition is satisfied, the pair $(b, a) \in A^2$ in a C^*-algebra A will be called *invertible* if there are v and u in A for which $vb + au = 1$, and will be called *weakly invertible* if there is implication, for arbitrary $c \in A$, $bc = c^*a = 0 \implies c = 0$.

Suppose $a = (a_1, a_2)$ is a pair of elements in a C^*-algebra A: then its *Koszul complex* $\Lambda(a)$ can be written by the matrices ([3] p.75; [4], (11.9.1.6),(11.9.2.5))

$$(0.1) \qquad 0 \longrightarrow A \xrightarrow{\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}} \begin{bmatrix} A \\ A \end{bmatrix} \xrightarrow{\begin{bmatrix} -a_2 & a_1 \end{bmatrix}} A \longrightarrow 0,$$

while $\Lambda(a)$ can be represented by the single matrix ([4],(11.9.1.7))

$$(0.2) \qquad \begin{bmatrix} 0 & 0 & 0 & 0 \\ a_1 & 0 & 0 & 0 \\ a_2 & 0 & 0 & 0 \\ 0 & -a_2 & a_1 & 0 \end{bmatrix} : \begin{bmatrix} A \\ A \\ A \\ A \end{bmatrix} \longrightarrow \begin{bmatrix} A \\ A \\ A \\ A \end{bmatrix}.$$

Each matrices in the representations (0.1) and (0.2) must be interpreted as a multiplication operator. If $a = (a_1, a_2) \in A^2$ is a commuting pair then $\Lambda(a)^2 = 0$. We shall write $\Lambda_2(A)$ for the set of 4×4 matrices whose entries are in A. Noting that $\Lambda_2(A)$ is also a C^*-algebra, the pair $a = (a_1, a_2) \in A^2$ can be classified as "Taylor nonsingular"(cf.[1],[3],[4],[6]): the pair $a = (a_1, a_2)$ will be called *Taylor invertible* with respect to

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A if there are U and V in $\Lambda_2(A)$ for which $V\Lambda(a) + \Lambda(a)U = I$ (this is equivalent to the condition of exactness of the chain complex (0.1)) and will be called *weakly Taylor invertible* with respect to A if there is implication, for arbitrary W in $\Lambda_2(A)$,

$$\Lambda(a)W = W^*\Lambda(a) = 0 \Longrightarrow W = 0.$$

Evidently, if $a = (a_1, a_2) \in A^2$ is a commuting pair then

$$a \in A^2 \text{ Taylor invertible} \Longrightarrow a \in A^2 \text{ weakly Taylor invertible.}$$

We recall that ([4], Theorem 11.10.6)

(0.3) $a \in A^2$ Taylor invertible $\Longleftrightarrow \Lambda(a)^*\Lambda(a) + \Lambda(a)\Lambda(a)^*$ invertible.

The weak invertibility for a pair can be tested by the zero divisorness of a single elements:

Lemma 1. *If $(b, a) \in A^2$ then*

(b, a) weakly invertible $\Longleftrightarrow$ $b^*b + aa^*$ not a zero divisor.

Proof. This is from ([6], Theorem 2).

We now have the analogue of (0.3) for the weakly Taylor invertibility:

Lemma 2. *If $a = (a_1, a_2) \in A^2$ is a commuting pair of elements in a C^*-algebra A then the followings are equivalent:*

(2.1) *$a \in A^2$ is weakly Taylor invertible with respect to A*
(2.2) *$\Lambda(a)^*\Lambda(a) + \Lambda(a)\Lambda(a)^*$ is not a zero-divisor with respect to $\Lambda_2(A)$*
(2.3) *$\Lambda(a) + \Lambda(a)^*$ is not a zero-divisor with respect to $\Lambda_2(A)$.*

Proof. The equivalence of (2.1) and (2.2) is by application of Lemma 1 to the chain $(\Lambda(a), \Lambda(a))$. The equivalence of (2.2) and (2.3) follows from

$$(\Lambda(a) + \Lambda(a)^*)^2 = \Lambda(a)^*\Lambda(a) + \Lambda(a)\Lambda(a)^*$$

which holds because $(\Lambda(a), \Lambda(a))$ is a chain.

Given a commuting pair $T = (T_1, T_2) \in B(H)^2$ for a Hilbert space H, it is well known that if T is Taylor invertible then the four operators $T_1^*T_1 + T_2^*T_2, T_1T_1^* + T_2T_2^*, T_1T_1^* + T_2^*T_2$ and $T_1^*T_1 + T_2T_2^*$ are invertible ([3], Proposition 3.7): This, in fact, holds for a commuting pair $(a_1, a_2) \in A^2$ in a C^*-algebra A. Can we say an analogue for the weakly Taylor invertibility in a C^*-algebra ?

THEOREM 3. *Let $a = (a_1, a_2) \in A^2$ be a commuting pair of elements in a C^*-algebra A. If $a \in A^2$ is weakly Taylor invertible then $a_1^* a_1 + a_2^* a_2, a_1 a_1^* + a_2 a_2^*, a_1 a_1^* + a_2^* a_2$ and $a_1^* a_1 + a_2 a_2^*$ are not zero-divisors.*

Proof. Suppose $a = (a_1, a_2)$ is weakly Taylor invertible. Then by Lemma 2

$$\text{(3.1)} \quad \Lambda(a)^*\Lambda(a) + \Lambda(a)\Lambda(a)^* = \begin{bmatrix} a_1^*a_1 + a_2^*a_2 & 0 & 0 & 0 \\ 0 & a_1a_1^* + a_2^*a_2 & a_1a_2^* - a_2^*a_1 & 0 \\ 0 & a_2a_1^* - a_1^*a_2 & a_1^*a_1 + a_2a_2^* & 0 \\ 0 & 0 & 0 & a_1a_1^* + a_2a_2^* \end{bmatrix}$$

is not a zero-divisor with respect to $\Lambda_2(A)$. Since a block diagonal matrix is not a zero-divisor if and only if each block is not a zero-divisor ([5], Theorem 1.3), $a_1^*a_1 + a_2^*a_2$ and $a_1a_1^* + a_2a_2^*$ are at least not a zero-divisor. For the other half rests, instead of representing the Koszul complex of (0.1) into the matrix of (0.2), we can put together its "odd"and "even"parts using the 'base change'([4], (11.10.7.9)):

$$\Lambda(a) = \begin{bmatrix} 0 & 0 & a_1 & 0 \\ 0 & 0 & a_2 & 0 \\ 0 & 0 & 0 & 0 \\ -a_2 & a_1 & 0 & 0 \end{bmatrix}.$$

By Lemma 2,

$$\Lambda(a) + \Lambda(a)^* = \begin{bmatrix} 0 & 0 & a_1 & -a_2^* \\ 0 & 0 & a_2 & a_1^* \\ a_1^* & a_2^* & 0 & 0 \\ -a_2 & a_1 & 0 & 0 \end{bmatrix}$$

is not a zero-divisor, and so is not $\begin{bmatrix} a_1^* & a_2^* \\ -a_2 & a_1 \end{bmatrix}$. It thus follows that for arbitrary $c \in A$,

$$\begin{bmatrix} a_1^* & a_2^* \\ -a_2 & a_1 \end{bmatrix} \begin{bmatrix} c & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Longrightarrow c = 0,$$

so that

$$a_1^* c = c^* a_2^* = 0 \Longrightarrow c = 0,$$

which says that (a_1^*, a_2^*) is weakly invertible with respect to A; therefore Lemma 1 says that $a_1 a_1^* + a_2^* a_2$ is not a zero-divisor. The argument for $a_1^* a_1 + a_2 a_2^*$ is the same.

We have the analogues of ([1], Corollaries 3.7 and 3.9):

COROLLARY 4. *A doubly commuting pair $a = (a_1, a_2) \in A^2$ (i.e., $[a_i, a_j^*] = 0$ for all $i \neq j$) is weakly Taylor invertible if and only if $a_1^* a_1 + a_2^* a_2, a_1 a_1^* + a_2 a_2^*, a_1 a_1^* + a_2^* a_2$ and $a_1^* a_1 + a_2 a_2^*$ are not zero-divisors.*

Proof. This at once follows from (3.1).

COROLLARY 5. *If a_1 and a_2 are normal and they commute then $a = (a_1, a_2)$ is weakly Taylor invertible if and only if $a_1^* a_1 + a_2^* a_2$ is not a zero-divisor.*

Proof. This is obvious from Corollary 4 and Fuglede-Putnam Theorem.

All our result in this note can be extended to the commuting n-tuple of elements in a C^*-algebra. The argument is clear from the case $n = 2$.

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