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TAYLOR INVERTIBILITY FOR A COMMUTING PAIR OF OPERATORS

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Throughtout this note, we suppose that H is a Hilbert space, and we write $B(H)$ for the set of all the bounded linear operators on H and T^* for the Hilbert adjoint operator of T. Let $T = (T_1, \dots, T_n)$ be a commuting n-tuple of operators acting on H, let $\Lambda(e) = \{\Lambda^k[e_1, \dots, e_n]\}_{n=0}^k$ be the exterior algebra on n generators ($e_i \wedge e_j = -e_j \wedge e_i$ for all $i, j = 1, \dots, n$), and write $\Lambda(T) : \Lambda[e]\otimes\mathrm{H} \longrightarrow \Lambda[e]\otimes\mathrm{H}$ given by

$$\Lambda(T)(w \otimes x) = \sum_{i=1}^{n} (e_i \wedge w) \otimes T_i x \tag{0.1}$$

for the *Koszul complex* of T. If $n = 2$ the Koszul complex $\Lambda(T)$ can be represented by the operator matrices

$$0 \longrightarrow H \xrightarrow{\begin{bmatrix} T_1 \\ T_2 \end{bmatrix}} H \oplus H \xrightarrow{\begin{bmatrix} -T_2 & T_1 \end{bmatrix}} H \longrightarrow 0. \tag{0.2}$$

Clearly $\Lambda(T)^2 = 0$, so ran $\Lambda(T) \subseteq$ ker $\Lambda(T)$. If ker $\Lambda(T) =$ ran $\Lambda(T)$ then T is said to be *Taylor invertible*. When $n = 2$, $T = (T_1, T_2)$ is Taylor invertible if and only if $\ker T_1 \cap \ker T_2 = 0, \operatorname{ran} T_1 + \operatorname{ran} T_2 = H$, and (if $T_1 y_2 = T_2 y_1$ then $y_1 = T_1 x$ and $y_2 = T_2 x$ for some $x \in X$) (cf. [1], [3], [6], [7]).

Given a commuting pair $T = (T_1, T_2)$, it is well known that if T is Taylor invertible then the four operators $T_1^* T_1 + T_2^* T_2$, $T_1 T_1^* + T_2 T_2^*$, $T_1 T_1^* + T_2^* T_2$ and $T_1^* T_1 + T_2 T_2^*$ are all invertible (see [2, Proposition 3.7]). However, the invertibility of those four operators does not guarentee, in general, the Taylor invertibility of T (for example, see [3, Application 5.30]).

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Now we can ask: What else, in addition to the invertibility of those four operators, is needed for the Taylor invertibility of $T = (T_1, T_2)$? R. E. Curto gave an answer in [2, Theorem 3.10], in which the Taylor invertibility of T was described in terms of the invertibility of $T_1^*T_1 + T_2^*T_2$ and $T_1T_1^* + T_2T_2^*$. In this note we shall give an another answer. For brevity, write

$$\widehat{T} = \begin{bmatrix} T_1 & -T_2^* \\ T_2 & T_1^* \end{bmatrix}.$$

Recall that T is Taylor invertibility if and only if $\widehat{T}$ is invertible (see [1, Theorem 1], in which the Koszul complex was constructed by the descending form).

We now meet:

THEOREM 1. *Let $T = (T_1, T_2)$ be a commuting pair of operators on a Hilbert space H and write $A = T_1^*T_1 + T_2^*T_2$, $B = T_1T_1^* + T_2T_2^*$, $C = T_1T_1^* + T_2^*T_2$, $D = T_1^*T_1 + T_2T_2^*$ and $G = T_1T_2^* - T_2^*T_1$. If A,B,C, and D are all invertible then T is Taylor invertible whenever one of the following conditions is satisfied:*

(a) $\|G\| < \sqrt{\gamma(C)\gamma(D)}$, *where $\gamma(K)$ denotes the minimum modulus of $K \in B(H)$.*

(b) *C commutes with $GD^{-1}G^*$ and $\|GD^{-1}G^*x\| \leq (1-\alpha)\|Cx\|$ for all $x \in H$ and for some $\alpha > 0$.*

Proof. We begin by showing that if A, B, C and D are all invertible then

$$(1.1) \qquad 1 - C^{-1}GD^{-1}G^* \text{ invertible} \Longrightarrow T \text{ Taylor invertible.}$$

To do this, observe that (cf. [6])

$$(1.2) \qquad \widehat{T}^*\widehat{T} = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \text{ invertible} \Longleftrightarrow \widehat{T} \text{ bounded below}$$

and

$$(1.3) \qquad \widehat{T}\widehat{T}^* = \begin{bmatrix} C & G \\ G^* & D \end{bmatrix} \text{ invertible} \Longleftrightarrow \widehat{T} \text{ open.}$$

Since A and B are both invertible, it follows from (1.2) that $\widehat{T}$ is bounded below. Now write

$$F = \begin{bmatrix} 1 - C^{-1}GD^{-1}G^* & 0 \\ D^{-1}G^* & 1 \end{bmatrix}.$$

If $1 - C^{-1}GD^{-1}G^*$ is invertible then F is invertible (cf. [4, Problem 71]). In particular, since

$$K = \begin{bmatrix} C & G \\ 0 & D \end{bmatrix} \text{ is also invertible,}$$

it follows that $\widehat{T}\widehat{T}^* = KF$ is invertible. Therefore by (1.3) $\widehat{T}$ is open, and hence invertible. This proves (1.1).

We now show that each condition of this theorem implies that $1 - C^{-1}GD^{-1}G^*$ is invertible.

(a) If $\|G\| < \sqrt{\gamma(C)\gamma(D)}$ then since $\gamma(C) = \|C^{-1}\|^{-1}$ and $\gamma(D) = \|D^{-1}\|^{-1}$ it follows that

$$\|C^{-1}GD^{-1}G^*\| \leq \|C^{-1}\| \, \|D^{-1}\| \, \|G\|^2 < 1;$$

therefore by the standard Banach algebra theory, $1 - C^{-1}GD^{-1}G^*$ is invertible.

(b) Observe that C is positive and

$$D \text{ positive} \Longrightarrow D^{-1} \text{ positive} \Longrightarrow GD^{-1}G^* \text{ positive.}$$

If $\|GD^{-1}G^*x\| \leq (1-\alpha)\|Cx\|$ for all $x \in H$ and for some $\alpha > 0$ this implies that $(1-\alpha)C - GD^{-1}G^*$ is positive because

$$\begin{aligned} \|GD^{-1}G^*x\|^2 \leq (1-\alpha)^2\|Cx\|^2 &\Rightarrow \langle (GD^{-1}G^*)^2x, x\rangle \leq (1-\alpha)^2\langle C^2x, x\rangle \\ &\Rightarrow (GD^{-1}G^*)^2 \leq ((1-\alpha)C)^2 \\ &\Rightarrow GD^{-1}G^* \leq (1-\alpha)C. \end{aligned}$$

The last implication comes from the fact that $GD^{-1}G^*$ and $(1-\alpha)C$ commute and are both positive. Remembering [5, Theorem 2.2] that if $A, B \in B(H)$ are positive then

$$A \text{ invertible} \Longrightarrow A + B \text{ invertible,}$$

we have $C - GD^{-1}G^* = \alpha C + ((1-\alpha)C - GD^{-1}G^*)$ is invertible, and hence so is $1 - C^{-1}GD^{-1}G^*$.

We might remark that if $T_1T_2^* - T_2^*T_1$ has sufficiently small norm or, in particular, if (T_1, T_2) is doubly commuting then, by the above theorem, (T_1, T_2) is Taylor invertible if and only if $T_1^*T_1 + T_2^*T_2$, $T_1T_1^* + T_2T_2^*$, $T_1T_1^* + T_2^*T_2$ and $T_1^*T_1 + T_2T_2^*$ are all invertible (cf. [1, Corollary 3.7]).

We conclude with:

THEOREM 2. *If $T = (T_1, T_2)$ is a doubly commuting pair of operators on a Hilbert space H then*

$$\partial\sigma_{\mathbf{T}}(T) \subseteq \sigma_{\mathbf{H}}(T) \cap \sigma_{\mathbf{H}}(\overline{T_1^*}, T_2),$$

where

$$\sigma_{\mathbf{T}}(T) := \left\{\lambda \in \mathbf{C}^2 : T - \lambda \text{ is not Taylor invertible}\right\},$$

$$\begin{aligned}\sigma_{\mathbf{H}}(T) := &\left\{\lambda \in \mathbf{C}^2 : (T_1-\lambda_1)^*(T_1-\lambda_1) + (T_2-\lambda_2)^*(T_2-\lambda_2)\right.\\ &\qquad \left.\text{is not invertible}\right\}\\ &\cup \left\{\lambda \in \mathbf{C}^2 : (T_1-\lambda_1)(T_1-\lambda_1)^* + (T_2-\lambda_2)(T_2-\lambda_2)^*\right.\\ &\qquad \left.\text{is not invertible}\right\},\end{aligned}$$

$$\partial K := \text{the topological boundary of } K \subseteq \mathbf{C}^2.$$

Proof. If $T = (T_1, T_2)$ is doubly commuting then from the argument of Theorem 1 and [6, (8.5.4.2)], we have

$$T_1^*T_1 + T_2^*T_2,\ T_1T_1^* + T_2T_2^* \text{ invertible} \Longleftrightarrow \widehat{T} \text{ bounded below}$$

and

$$\begin{aligned}T_1T_1^* + T_2^*T_2,\ T_1^*T_1 + T_2T_2^* \text{ invertible} &\Longleftrightarrow \widehat{T}^* \text{ bounded below}\\ &\Longleftrightarrow \widehat{T} \text{ open.}\end{aligned}$$

Remembering that if $S \in B(H)$ for a Hilbert space H, then by [6, Theorem 9.3.3] $\partial\sigma(S) \subseteq \sigma_{ap}(S) \cap \sigma_{ap}(S^*)$, where $\sigma_{ap}(S) = \{\lambda \in \mathbf{C} : S - \lambda \text{ is not bounded below}\}$, we have

$$\partial\sigma_{\mathbf{T}}(T) \subseteq \sigma_{\mathbf{H}}(T) \cap \sigma_{\mathbf{H}}(\overline{T_1^*}, T_2).$$

We cannot expect that the analogs of Theorem 1 and Corollary 2 hold for $n \geq 3$. For an example, see [2, Remarks 3.4].

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