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STABLE RANKS AND REAL RANKS OF C^*–ALGEBRAS

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For a unital C^*-algebra A, we denote the *real rank* of A to be the smallest integer, $RR(A)$, such that for each n-tuple $(x_1, x_2, \dots, x_n)$ of self-adjoint elements in A with $n \le RR(A)+1$, and every $\epsilon > 0$, there is an n-tuple $(y_1, y_2, \dots, y_n)$ of elements in A_{sa} such that $\sum y_k^2$ is invertible and $\| \sum (x_k - y_k)^2 \| \le \epsilon$. Identifying each n-tuple $(x_1, x_2, \dots, x_n)$ with the matrix x in $M_n(A)$ that has $x_1, x_2, \dots, x_n$ as its first column and zero's elsewhere, and similarly for y, the estimate simply means that $\|x - y\| \le \epsilon$ in $M_n(A)$. Moreover, the invertibility of $\sum {y_k}^2$ is equivalently expressed by the equation $1 = \sum z_k y_k$ for a suitable n-tuple $(z_1, z_2, \dots, z_n)$. For any C^*-algebra A with identity we denote by $Lg_n(A)$ the set of all n-tuples of A which generates A as a left ideal. By the *stable rank*, denoted $sr(A)$, we mean the least integer n such that $Lg_n(A)$ is dense in A^n for the product topology. If no such integer exists, we set $sr(A) = \infty$. If A has no identity element, then $sr(A)$ or $RR(A)$ are defined to be those of C^*-algebra $\tilde{A}$ obtained from A by adjoining an identity element. Note that the topological stable rank in [9] is the same as the stable rank for C^*-algebras ([6]).

Note that for a C^*-algebra A, $sr(A) = 1$ is equivalent to the fact that the set of invertible elements of A is dense in A and $RR(A) = 0$ is equivalent to the fact that the set of self-adjoint invertible elements of A is dense in A_{sa} (see [10],[11]).

In this note, we examine some properties about the stable ranks and real ranks of C^*-algebras, especially the cases $sr(A) = 1$ and $RR(A) = 0$, which seem to be the most tractable cases.

LEMMA 1([3]). *Suppose that A is a unital C^*-algebra, p is a projection in A and $x \in A$ such that the element $b = (1-p)x(1-p)$ is*

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invertible in $(1-p)A(1-p)$. *Then* x *is invertible if and only if* $a-cb^{-1}d$ *is invertible in* pAp, *where* $a=pxp, c=px(1-p)$ *and* $d=(1-p)xp$.

The following proposition was proved for real rank in [3].

PROPOSITION 2. *If* A *is a unital* C^**-algebra with* $sr(A)=1$, *then* $sr(pAp)=1$ *for every projection* p *in* A. *And if* $sr(pAp)=sr((1-p)A(1-p))=1$ *for some projection* p *in* A, *then* $sr(A)=1$.

Proof. Let $x \in pAp$ and $\epsilon > 0$ be given. Since $sr(A)=1$, there exists an invertible element $y \in A$ such that $\|x+1-p-y\| \leq \epsilon$. Letting $b=(1-p)y(1-p)$, we have $\|1-p-b\| = \|(1-p)(x+1-p-y)(1-p)\| \leq \epsilon$. Assuming $\epsilon < 1$, it follows that b is invertible in $(1-p)A(1-p)$. By lemma 1, it follows that $z = pyp - py(1-p)b^{-1}(1-p)yp$ is invertible in pAp. Since $\|b^{-1}\| = \|\sum_{n=0}^{\infty}[(1-p)-b]^n\| \leq \dfrac{1}{1-\epsilon}$, we have $\|py(1-p)b^{-1}(1-p)yp\| \leq \|py(1-p)\|\ \|b^{-1}\|\ \|(1-p)yp\| \leq \dfrac{\epsilon^2}{1-\epsilon}$. Thus $\|x-y\| = \|x-pyp+py(1-p)b^{-1}(1-p)yp\| \leq \epsilon + \dfrac{\epsilon^2}{1-\epsilon}$. This shows that the stable rank of pAp is equal to 1. Next, take $x \in A$ and write it as the obvious matrix notation $x = \begin{pmatrix} a & d \\ c & b \end{pmatrix}$. Given $\epsilon > 0$, we can take invertible $b_0 \in (1-p)A(1-p)$ such that $\|b-b_0\| \leq \epsilon$. Considering $a - bd_0^{-1}c \in pAp$, there exists an element $z \in pAp$ such that $\|z-(a-db_0^{-1}c)\| \leq \epsilon$. Let $y = z + db_0^{-1}c$. Then $z = y - db_o^{-1}c$ is invertible. Hence by lemma 1, $x_o = \begin{pmatrix} y & d \\ c & b_0 \end{pmatrix}$ is invertible. Note that $\|y-a\| = \|y-db_0^{-1}c-(a-db_0^{-1}c)\| = \|z-(a-db_0^{-1}c)\| \leq \epsilon$. Therefore $\|x-x_0\| = \| \begin{pmatrix} y-a & 0 \\ 0 & b-b_0 \end{pmatrix} \| \leq \epsilon$. This completes the proof.

COROLLARY 3. *If* A *and* B *are unital* C^**-algebras with* $sr(A\otimes B)=1$ *and* B *has a minimal projection, then* $sr(A)=1$.

PROPOSITION 4. *If a* C^**-algebra* A *is the inductive limit of a net* $(A_\lambda)_{\lambda\in\Lambda}$ *of* C^**-algebras with stable ranks* 1, *then* $sr(A)=1$.

Proof. Let $x \in A$ and $\epsilon > 0$ be given. There exists an element $x_\lambda \in A_\lambda$ such that $\|x - x_\lambda\| \leq \dfrac{\epsilon}{2}$ for some $\lambda \in \Lambda$. Since $sr(A_\lambda)=1$, we can find an invertible $y_\lambda \in A_\lambda$ such that $\|x_\lambda - y_\lambda\| \le \frac{\epsilon}{2}$. Thus $\|x - y_\lambda\| \le \epsilon$, completing the proof.

COROLLARY 5. *If A is a unital C^*-algebra with $sr(A) = 1$ and B is an AF-algebra, then $sr(A \otimes B) = 1$.*

There is no stably finite simple C^*-algebra known to have stable rank greater than one. Hence there arises a natural question: Does every stably finite simple C^*-algebra have the stable rank one? About this question, there is a partial result in [10]; If A is a unital simple C^*-algebra, and B is a UHF-algebra, then A is stably finite if and only if $sr(A \otimes B) = 1$.

PROPOSITION 6. *Let A be a unital simple C^*-algebra. Then A is stably finite if and only if pAp and $(1-p)A(1-p)$ are stably finite for some projection $p \in A$.*

Proof. Let B be a UHF-algebra. Since A is stably finite if and only if $sr(A \otimes B) = 1$, $sr[(p \otimes 1)(A \otimes B)(p \otimes 1)] = sr[(1-p) \otimes 1](A \otimes B)[(1-p) \otimes 1] = 1$ if and only if pAp and $(1-p)A(1-p)$ are stably finite.

COROLLARY 7. *Let A be a unital simple C^*-algebra. Then A is stably finite if and only if $M_2(A)$ is stably finite.*

A hereditary C^*-subalgebra H of a C^*-algebra A is said *full* if the norm closure of AHA is equal to A. Note that any hereditary C^*-subalgebra of a simple C^*-algebra is full. For positive $x \in A$, denote by A_x the hereditary C^*-subalgebra of A generated by x. An element x of a unital C^*-algebra A is said to be *well-supported* if there is a projection $p \in A$ with $x = xp$ and x^*x is invertible in pAp. Recall that x is well-supported if and only if A_x is unital ([1]).

REMARK 8. Blackadar ([1]) has made the following conjecture : "Let A be a C^*-algebra and if B is an arbitrary full hereditary C^*-subalgebra of A, then $sr(B) \ge sr(A)$."

The following proposition shows that this is not the case.

PROPOSITION 9. *In the Cuntz algebra O_n, there is a hereditary C^*-subalgebra A_x such that $sr(A_x) < sr(O_n)$.*

Proof. Let A be O_n. If we modify the proof of theorem 6 in [7], not every positive element of A is well-supported. Let x be a nonzero positive element which is not well-supported. Since O_n is purely infinite, A_y is either unital or stable for every nonzero positive $y \in A$, so that we have A_x is a stable C^*-subalgebra. Since $sr(A_x) = 1$ or 2 by [9] and $sr(A) = \infty$, we have the conclusion.

Recall that S. Zhang ([11]) showed that for a simple C^*-algebra A, $RR(A) = 0$ and every nonzero projection is infinite if and only if for every positive element x in A, there exists an infinite projection in A_x.

Let A be a C^*-algebra and $a, b \in A^+$. We write $a \approx b$ if there is a set of elements $\{x_i\}$ in A such that $a = \sum x_i^* x_i$ and $b = \sum x_i x_i^*$. An element a in A^+ is said to be '$\approx$-finite' if $b \leq a$ and $a \approx b$ implies $a = b$. In [8], they used this equivalence relation to compare the positive elements of C^*-algebras as F. J. Murray and J. von Neumann introduced the well-known notion of equivalence between projections of von Neumann algebras. The following proposition generalizes the fact that every type III factor is purely infinite.

PROPOSITION 10. *Let A be a unital monotone closed infinite simple C^*-algebra such that 'finite' implies '$\approx$-finite' when restricted to the set of projections in A. Then $RR(A) = 0$ if and only if A is purely infinite.*

Proof. Assume that there is a finite projection p in A. By Zhang ([12]), there exist mutually orthogonal projections $r_1, \cdots, r_n$ in A such that $1 - p = \sum r_i$ and $r_1 \preceq r_2 \preceq \cdots \preceq r_n \preceq p$. So there exist projections $q_1, \cdots, q_n$ in A such that $r_i \sim q_i \leq p$. We claim each q_i is a finite projection. Suppose that q_i is infinite. Then there exists x such that $xx^* = q_i' < q_i = x^*x \leq p$. Then letting $\alpha = p - q_i$, $(q_i' x q_i + \alpha)^*(q_i' x q_i + \alpha) = q_i x^* q_i' x q_i + \alpha = q_i x^* x x^* x q_i + \alpha = q_i + \alpha = p$ and $(q_i' x q_i + \alpha)(q_i' x q_i + \alpha)^* = q_i' x q_i x^* q_i' + \alpha = q_i' x x^* x x^* q_i' + \alpha = q_i' + \alpha = p - (q_i - q_i') < p$. This is a contradiction to the fact that p is finite. Suppose that $r_i \sim r_0 \leq r_i$ for some r_o. Then there exist v, w in A such that $v^*v = q_i, vv^* = r_i, w^*w = r_i$ and $ww^* = r_o$. Hence $(v^*wv)^*(v^*wv) = q_i$ and $(v^*wv)(v^*wv)^* = v^* r_0 v \leq q_i$. Since q_i is finite, $v^* r_0 v = q_i$. Hence $vv^* r_0 vv^* = r_i v_0 r_i = r_0 = v q_i v^* = vv^* = r_i$. This shows that r_i is also finite. Since A is monotone closed, the sum of finitely many finite projections is finite by [8]. This is a contradiction and there are no finite projections in A. Hence A is purely infinite, i.e. A_x has an infinite projection for every $x \in A^+$. The converse is clear ([3]).

Given $\epsilon > 0$, let

$$f_\epsilon(t) = \begin{cases} 0 & \text{on} \quad 0 \le t \le \dfrac{\epsilon}{2}, \\ \dfrac{2}{\epsilon}t - 1 & \text{on} \quad \dfrac{\epsilon}{2} < t < \epsilon, \\ 1 & \text{on} \quad t \ge \epsilon. \end{cases}$$

PROPOSITION 11. *Let A be a unital simple C^*-algebra. Then $RR(A) = 0$ if and only if $(A \otimes O_n)_{f_\delta(x)}$ has a nonzero projection for any x with $f_\delta(x) \neq 0, (x > 0, \delta > 0)$, where O_n is the Cuntz algebra.*

Proof. Let $\phi_n : O_n \to O_n$ be an endomorphism such that $\phi_n(x) = \sum_1^n s_i x s_i^*$. By [5], ϕ_n is homotopic to the identity map; i.e. there is a continuous path $\eta : [0,1] \to End(O_n)$ such that η_0=id and $\eta_1 = \phi_n$. Consider $1 \otimes \phi_n : A \otimes O_n \to A \otimes O_n$. Then any nonzero projection p is equivalent to $\sum(1 \otimes s_i)p(1 \otimes s_i)^*$. Note that p is equivalent to $(1 \otimes s_i)p(1 \otimes s_i)^*$. This shows that p is an infinite projection. Therefore the 'if' part is proved by [2] and the converse is clear.

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