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SOME PROPERTIES OF CONVOLUTION OPERATORS IN THE CLASS $\mathcal{P}_\alpha(\beta)$

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Making use of several families of convolution operators, we introduce and study a certain general class $\mathcal{P}_\alpha(\beta)(0 \le \alpha < 1; \beta \ge 0)$ of analytic functions in the open unit disk $\mathcal{U}$. We also investigate the relationships between the class $\mathcal{P}_\alpha(\beta)$ and the Hardy space $\mathcal{H}^\infty$ (of bounded analytic functions in $\mathcal{U}$). Finally, we consider some interesting applications of the results presented here to a class of generalized hypergeometric functions.

1. Introduction and definitions

Let $\mathcal{A}$ denote the class of (*normalized*) functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the *open* unit disk

$$\mathcal{U} = \{z : |z| < 1\}.$$

We also denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions which are univalent in $\mathcal{U}$.

A function $f(z) \in \mathcal{A}$ is said to be in the class $\mathcal{P}_\alpha(0 \le \alpha < 1)$ if and only if it satisfies the inequality:

$$Re\{f'(z)\} > \alpha \quad (0 \le \alpha < 1; z \in \mathcal{U}).$$

The class $\mathcal{P}_0$ was investigated systematically by MacGregor [8] who did refer to numerous earlier studies involving functions whose derivative

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has a positive real part. Indeed, as readily implied by the Noshiro-Warschawski theorem (cf. e.g., Duren [3, p.47, Theorem 2.16]), $\mathcal{P}_\alpha$ is a subclass of the class $\mathcal{S}$.

Let f and g be in the class $\mathcal{A}$, with $f(z)$ given by (1.1), and $g(z)$ by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n. \tag{1.3}$$

For a given $f \in \mathcal{A}$, we define the convolution operator

$$\Omega_f : \mathcal{A} \to \mathcal{A}$$

by

$$\Omega_f(g) = f * g, \tag{1.4}$$

where, as usual, $f * g$ denotes the Hadamard product of f and g:

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n. \tag{1.5}$$

For a function $f \in \mathcal{A}$ given by (1.1), Owa and Srivastava ([11]; see also [12, p.338]) defined the generalized Libera integral operator $\mathcal{F}_c$ by

$$\begin{aligned} \mathcal{F}_c(f) &= \frac{c+1}{z^c} \int_0^z t^{c-1} f(t) dt (c > -1) \\ &= z + \sum_{n=2}^{\infty} \frac{c+1}{c+n} a_n z^n. \end{aligned} \tag{1.6}$$

The operator $\mathcal{F}_c$, when $c \in \mathrm{N} = \{1, 2, 3, \dots\}$, was introduced by Bernardi [1]. In particular, the operator $\mathcal{F}_1$ was studied earlier by Libera [6] and Livingston [7].

Clearly, (1.6) yields

$$f(z) \in \mathcal{A} \Rightarrow \mathcal{F}_c(f) \in \mathcal{A} \quad (c > -1). \tag{1.7}$$

Thus, we define $\mathcal{F}_c$ by

$$\mathcal{F}_c^n(f) = \begin{cases} \mathcal{F}_c \mathcal{F}_c^{n-1}(f) & (n \in \mathrm{N}), \\ f(z) & (n = 0). \end{cases} \tag{1.8}$$

With a view to introducing an interesting generalization of the class $\mathcal{P}_\alpha$, we now recall the following definition of a *multiplier transformation* (or *fractional integral* and *fractional derivative*):

DEFINITION 1. (Flett [4, p.748]). Let the function

$$\phi(z) = \sum_{n=0}^{\infty} c_n z^n \quad (z \in \mathcal{U}) \tag{1.9}$$

be analytic in $\mathcal{U}$ and let λ be a real number. Then the *multiplier transformation* $I^\lambda \phi$ is defined by

$$I^\lambda \phi(z) = \sum_{n=0}^{\infty} (n+1)^{-\lambda} c_n z^n \quad (z \in \mathcal{U}). \tag{1.10}$$

The function $I^\lambda \phi$ is clearly analytic in $\mathcal{U}$. It may be regarded as a *fractional integral* (for $\lambda > 0$) or *fractional derivative* (for $\lambda < 0$) of ϕ, and it is readily seen that

$$I^\lambda I^\mu \phi = I^{\lambda+\mu} \phi$$

for all real numbers λ and μ. Furthermore, in terms of the Gamma function, we have

$$\begin{aligned} I^\lambda \phi(z) &= \frac{1}{z\Gamma(\lambda)} \int_0^z \left[\log \frac{z}{t}\right]^{\lambda-1} \phi(t)dt \quad (\lambda > 0) \\ &= \frac{1}{\Gamma(\lambda)} \int_0^1 \left[\log \frac{1}{t}\right]^{\lambda-1} \phi(zt)dt \quad (\lambda > 0), \end{aligned} \tag{1.12}$$

which can be verified fairly easily by term-by-term integration, using some well-known Γ-function integrals.

Definition 1 leads us naturally to

DEFINITION 2. The *fractional derivative* $D^\lambda \phi$ of order $\lambda \geq 0$, for an analytic function ϕ given by (1.9), is defined by

$$D^\lambda \phi(z) = I^{-\lambda} \phi(z) = \sum_{n=0}^{\infty} (n+1)^\lambda c_n z^n (\lambda \geq 0; z \in \mathcal{U}). \tag{1.13}$$

It follows from Definition 2 that

$$D^m \phi(z) = \left[\frac{d}{dz} z\right]^m \phi(z) \quad (m \in \mathbf{N}_0 = \mathbf{N} \cup \{0\}). \tag{1.14}$$

More importantly, making use of Definition 2, we now introduce an interesting generalization of the class $\mathcal{P}_\alpha$ of functions in $\mathcal{A}$ which satisfy the inequality (1.2).

DEFINITION 3. A function $f(z) \in \mathcal{A}$ is said to be in the class $\mathcal{P}_\alpha(\beta)$ $(0 \leq \alpha < 1; \beta \geq 0)$ if and only if

$$2^{-\beta} D^\beta f \in \mathcal{P}_\alpha \quad (0 \leq \alpha < 1; \beta \geq 0).$$

Observe that $\mathcal{P}_\alpha(0) = \mathcal{P}_\alpha$. Furthermore, since $f \in \mathcal{A}$, it follows from (1.1) and (1.13) that

$$2^{-\beta} D^\beta f(z) = z + \sum_{n=2}^{\infty} [\frac{n+1}{2}]^\beta a_n z^n \quad (z \in \mathcal{U}), \tag{1.15}$$

which shows that $2^{-\beta} D^\beta f \in \mathcal{A}$ if $f \in \mathcal{A}$.

The object of the present paper is to investigate various useful properties of the general class $\mathcal{P}_\alpha(\beta)$ by using such families of convolution operators as those mentioned above. We also relate the class $\mathcal{P}_\alpha(\beta)$ with the Hardy space $\mathcal{H}^\infty$ of bounded analytic functions in $\mathcal{U}$, and consider several applications of our results to a class of generalized hypergeometric functions.

2. A Preliminary Lemma

In our present investigation of the general class $\mathcal{P}_\alpha(\beta)(0 \leq \alpha < 1; \beta \geq 0)$, we shall require the following

LEMMA (MILLER AND MOCANU [9, P.301, THEOREM 10]). *Let $M(z)$ and $N(z)$ be analytic in $\mathcal{U}$ with*

$$M(0) = N(0) = 0, \tag{2.1}$$

and let γ be a real number. If $N(z)$ maps $\mathcal{U}$ onto a (possibly many-sheeted) region which is starlike with respect to the origin, then

$$Re\left\{\frac{M'(z)}{N'(z)}\right\} > \gamma(z \in \mathcal{U}) \Rightarrow Re\left\{\frac{M(z)}{N(z)}\right\} > \gamma(z \in \mathcal{U}) \tag{2.2}$$

and

$$Re\left\{\frac{M'(z)}{N'(z)}\right\} < \gamma(z \in \mathcal{U}) \Rightarrow Re\left\{\frac{M(z)}{N(z)}\right\} < \gamma(z \in \mathcal{U}). \tag{2.3}$$

3. Examples of Convolution Operators with Integral Representations

Throughout this section, let $f(z) \in \mathcal{A}$ be given by (1.1). Suppose also that [cf. Equations (1.6) and (1.8)]

$$\begin{aligned} \mathcal{T}_p(f) &= \mathcal{F}_{c_1} \cdots \mathcal{F}_{c_p}(f) \\ &= z + \sum_{n=2}^{\infty} \frac{(c_1+1)\cdots(c_p+1)}{(c_1+n)\cdots(c_p+n)} a_n z^n \\ &(c_j > -1 (j = 1, \cdots, p); p \in \mathbf{N}). \end{aligned} \tag{3.1}$$

Then, in view of the definitions (1.5) and (1.6), it is not difficult to express the functional $\mathcal{T}_p$ as a convolution operator given by

$$\mathcal{T}_p(f) = \mathcal{F}_{c_1}\left[\frac{z}{1-z}\right] * \cdots * \mathcal{F}_{c_p}\left[\frac{z}{1-z}\right] * f. \tag{3.2}$$

For various special choices for the parameters $c_j (j = 1, \cdots, p)$, the function $\mathcal{T}_p(f)$ can be simplified considerably, giving us some (single) integral representations which are contained in the following examples.

EXAMPLE 1. Setting

$$c_j = j + \gamma \quad (\gamma > -2; j = 1, \cdots, p)$$

in (3.1), we obtain

$$\mathcal{T}_p(f) = \{B(p, \gamma+2)\}^{-1} \int_0^1 t^\gamma (1-t)^{p-1} f(zt) dt (r > -2; p \in \mathbf{N}) \tag{3.3}$$

or, equivalently,

(3.4)

$$\mathcal{T}_p(f) = \{z^{\gamma+1}, B(p, \gamma+2)\}^{-1} \int_0^z t^\gamma \left[1 - \frac{t}{z}\right]^{p-1} f(t) dt (r > -2; p \in \mathbf{N}),$$

where $B(\alpha, \beta)$ denotes the Beta function defined by

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

EXAMPLE 2. For $\gamma = r(r \in \mathbf{N}_0)$, the last integral representation (3.4) can be written in the form

$$(3.5) \qquad \mathcal{T}_p(f) = \frac{(p+r+1)!}{(p-1)!(r+1)!} z^{-r-1} \int_0^z t^r \left[1 - \frac{t}{z}\right]^{p-1} f(t)dt \qquad (r \in \mathbf{N}_0; p \in \mathbf{N}),$$

which, for $r = 0$, was given by Bernardi [1, p. 438, Example 3].

EXAMPLE 3. Setting $c_j = 1 (j = 1, \ldots, p)$ in (3.1), and making use of (1.12), we have

$$(3.6) \qquad \begin{aligned} \mathcal{T}_p(f) &= \mathcal{F}_1^p(f) = 2^p I^p f(z) \\ &= \frac{2^p}{(p-1)!} z^{-1} \int_0^z \left[\log \frac{z}{t}\right]^{p-1} f(t)dt (p \in \mathbf{N}). \end{aligned}$$

4. Inclusion Properties of the General Class $\mathcal{P}_\alpha(\beta)$

We begin by stating a generalization of an interesting result due to Bernardi [1, p. 432, Theorem 4] as

THEOREM 1. *Let the function $f(z)$ be in the class $\mathcal{P}_\alpha(\beta)$. Then $\mathcal{F}_c(f)$ defined by* (1.6) *is also in the class $\mathcal{P}_\alpha(\beta)$.*

Proof. A simple calculation shows that

$$(4.1) \qquad \frac{d}{dz} D^\beta(\mathcal{F}_c(f)) = \frac{c+1}{z^{c+1}} \int_0^z t^c \left\{\frac{d}{dt} D^\beta f(t)\right\} dt,$$

where the operators $\mathcal{F}_c (c > -1)$ and $D^\lambda (\lambda \geq 0$ are defined by (1.6) and (1.13), respectively. In view of (4.1), we set

$$(4.2) \qquad M(z) = \frac{c+1}{2^\beta} \int_0^z t^c \left\{\frac{d}{dt} D^\beta f(t)\right\} \text{ and } N(z) = z^{c+1},$$

so that

$$(4.3) \qquad Re\left\{\frac{M'(z)}{N'(z)}\right\} = Re\left\{2^{-\beta} \frac{d}{dz}(D^\beta f(z))\right\}.$$

Since, by hypothesis, $f \in \mathcal{P}_\alpha(\beta)$, the second member of (4.3) is greater than $\alpha(z \in \mathcal{U})$, and hence

$$\text{(4.4)} \qquad Re\left\{\frac{M'(z)}{N'(z)}\right\} > \alpha(0 \leq \alpha < 1; z \in \mathcal{U}).$$

Thus, applying the lemma of Section 2, we have

$$\text{(4.5)} \qquad Re\left\{\frac{M(z)}{N(z)}\right\} = Re\left\{2^{-\beta}\frac{d}{dz}D^\beta(\mathcal{F}_c(f))\right\}\alpha$$
$$(0 \leq \alpha < 1; \beta \geq 0; z \in \mathcal{U}),$$

which evidently completes the proof of Theorem 1.

REMARK 1. It follows from the definitions (1.6) and (1.13) that

$$\text{(4.6)} \qquad 2^{-\beta}D^\beta(\mathcal{F}_c(f)) = \mathcal{F}_c(2^{-\beta}D^\beta f)(c > -1; \beta \geq 0; f \in \mathcal{A}),$$

which can be used to give an alternative proof of Theorem 1 along the lines of Bernardi [1, p.432].

In conjunction with the first part of the definition (3.1), Theorem 1 readily yields.

COROLLARY 1. *Let the function $f(z)$ be in the class $\mathcal{P}_\alpha(\beta)$. Then the function $\mathcal{T}_p(f)$ defined by (3.1) is also in the class $\mathcal{P}_\alpha(\beta)$.*

The next inclusion property of the class $\mathcal{P}_\alpha(\beta)$, contained in Theorem 2 below, would involve the operator $\mathcal{F}_1^\lambda(\lambda > 0)$ defined by

$$\text{(4.7)} \qquad \mathcal{F}_1^\lambda(f) = 2^\lambda I^\lambda f(z) \quad (\lambda > 0; f \in \mathcal{A}),$$

which, for $\lambda = p \in \mathbf{N}$, was considered already in (3.6). Clearly, we have

$$\text{(4.8)} \qquad f(z) \in \mathcal{A} \Rightarrow \mathcal{F}_1^\lambda(f) \in \mathcal{A} \quad (\lambda > 0).$$

THEOREM 2. *Let the function $f(z)$ be in the class $\mathcal{P}_\alpha(\beta)$. Then the function $\mathcal{F}_1^\lambda(f)(\lambda > 0)$ defined by (4.7) is also in the class $\mathcal{P}_\alpha(\beta)$.*

Proof. Making use of (1.10) and (1.13), the definition (4.7) immediately yields [cf. Equation (4.6)]

$$2^{-\beta}D^\beta(\mathcal{F}_1^\lambda(f)) = \mathcal{F}_1^\lambda(2^{-\beta}D^\beta f)(\beta \geq 0; \lambda > 0; f \in \mathcal{A}). \tag{4.9}$$

Therefore, setting

$$g(z) = 2^{-\beta}D^\beta f \text{ and } G(z) = \mathcal{F}_1^\lambda(g), \tag{4.10}$$

we must show that

$$Re\{G'(z)\} > \alpha \quad (0 \leq \alpha < 1; z \in \mathcal{U}) \tag{4.11}$$

whenever $f \in \mathcal{P}_\alpha(\beta)$.

From the second integral representation in (1.12), we obtain

$$G'(z) = \frac{2^\lambda}{\Gamma(\lambda)}\int_0^1 \left[\log\frac{1}{t}\right]^{\lambda-1} tg'(zt)dt(\lambda > 0), \tag{4.12}$$

so that

$$Re\{G'(z)\} = \frac{2^\lambda}{\Gamma(\lambda)}\int_0^1 \left[\log\frac{1}{t}\right]^{\lambda-1} tRe\{g'(zt)\}dt(\lambda > 0), \tag{4.13}$$

Since $f \in \mathcal{P}_\alpha(\beta)$, we have

$$Re\{g'(zt)\} > \alpha(0 \leq \alpha < 1; z \in \mathcal{U}; 0 \leq t \leq 1), \tag{4.14}$$

and hence (4.13) yields

$$Re\{G'(z)\} > \frac{2^\lambda}{\Gamma(\lambda)}\alpha\int_0^1 \left[\log\frac{1}{t}\right]^{\lambda-1} tdt = \alpha(0 \leq \alpha < 1; \lambda > 0, \tag{4.15}$$

which completes the proof of Theorem 2.

COROLLARY 2. *If* $0 \le \alpha < 1$ *and* $0 \le \beta < \gamma$, *then* $\mathcal{P}_\alpha(\gamma) \subset \mathcal{P}_\alpha(\beta)$.

Proof. Setting $\lambda = \gamma - \beta > 0$ in Theorem 2, we observe that

$$\begin{aligned} f(z) \in \mathcal{P}_\alpha(\gamma) &\Rightarrow \mathcal{F}_1^{\gamma-\beta}(f) \in \mathcal{P}_\alpha(\gamma) \\ &\Leftrightarrow \{2^{-\gamma} D^\gamma(\mathcal{F}_1^{\gamma-\beta}(f))\} \in \mathcal{P}_\alpha \\ &\Leftrightarrow 2^{-\beta} D^\beta f \in \mathcal{P}_\alpha \\ &\Leftrightarrow f \in \mathcal{P}_\alpha(\beta), \end{aligned} \tag{4.16}$$

and the proof of Corollary 2 is completed.

Next we define a function $h(z) \in \mathcal{A}$ by

$$h(z) = \sum_{n=1}^{\infty} [\frac{n+1}{2}] z^n = \frac{z - \frac{1}{2}z^2}{(1-z)^2} (z \in \mathcal{U}). \tag{4.17}$$

Then, in terms of the convolution operator Ω_f defined by (1.4), we have

$$\Omega_h(f) = (h * f)(z) = \frac{1}{2}\{f(z) + zf'(z)\} \quad (f \in \mathcal{A}), \tag{4.18}$$

which, when compared with (1.14) with $m = 1$, yields

$$\Omega_h(f) = \frac{1}{2} D^1 f \quad (f \in \mathcal{A}).$$

We now state and prove yet another inclusion property of the class $\mathcal{P}_\alpha(\beta)$, which is given by

THEOREM 3. *If* $0 \le \alpha < 1$ *and* $\beta \ge 0$, *then*

$$\mathcal{P}_\alpha(\beta + 1) \subset \mathcal{P}_\mu(\beta) \quad [\mu = \frac{4\alpha + 1}{5}]. \tag{4.20}$$

Proof. In view of (4.19) and Theorem 2 of Owa and Nunokawa [10, p.580], we have

$$\begin{aligned} f \in \mathcal{P}_\alpha(\beta + 1) &\Leftrightarrow 2^{-\beta-1} D^{\beta+1} f \in \mathcal{P}_\alpha \\ &\Rightarrow \Omega_h(2^{-\beta} D^\beta f) \in \mathcal{P}_\alpha \\ &\Rightarrow 2^{-\beta} D^\beta f \in \mathcal{P}_\mu [\mu = \frac{4\alpha + 1}{5}] \\ &\Leftrightarrow f \in \mathcal{P}_\mu(\beta) [\mu = \frac{4\alpha + 1}{5}], \end{aligned} \tag{4.21}$$

which evidently proves Theorem 3.

REMARK 2. Since $0 \leq \alpha < 1$, we have

$$\mu = \frac{4\alpha + 1}{5} > \alpha,$$

and hence $\mathcal{P}_\mu(\beta) \subset \mathcal{P}_\alpha(\beta)$.

REMARK 3. Since $\Omega_h(\mathcal{P}_0) \not\subset \mathcal{P}_0$, as observed by Livingston [7, p.356], we can apply the relationship (4.19) to conclude that $2^{-1}D^1 f$ need not be contained in $\mathcal{P}_0$ whenever $f \in \mathcal{P}_0$. Thus, by Definition 3,

$$\mathcal{P}_0(0) \not\subset \mathcal{P}_0(1).$$

5. Relationships with the Hardy Space

For a function f analytic in $\mathcal{U}$, we define the integral means by

$$M_p(r,f) = \begin{cases} \left[\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta\right]^{\frac{1}{p}} & (0 < p < \infty) \\ \mathcal{M}ax_{|z| \leq r}|f(z)| & (p = \infty). \end{cases} \tag{5.1}$$

The Hardy space $\mathcal{H}^p (0 < p \leq \infty)$ is the class of all functions f analytic in $\mathcal{U}$ for which

$$\lim_{r \to 1}\{M_p(r,f)\} < \infty \quad (0 < p \leq \infty). \tag{5.2}$$

For the general theory of $\mathcal{H}^p$ spaces, see (for example) Duren [2] and Koosis [5].

A simple relationship between the class $\mathcal{P}_\alpha(\beta)$ and the Hardy space $\mathcal{H}^p$ is given by

THEOREM 4. $\mathcal{P}_0(1) \subset \mathcal{H}^\infty$.

Proof. Suppose that $f \in \mathcal{P}_0(1)$. Then, by Definition 3, we have

$$Re\{(2^{-1}D^1 f)'\} > 0 \quad (z \in \mathcal{U}), \tag{5.3}$$

which, in view of a known result [2, p.34, Theorem 3.2], implies that

$$(2^{-1}D^1 f)' \in \mathcal{H}^p \quad (p < 1). \tag{5.4}$$

By the Hardy-Littlewood theorem [2, p.88, Theorem 5.12], (5.4) shows that $D^1 f \in \mathcal{H}^p$ for all $p < \infty$. Also, by Corollary 2, we have

$$f \in \mathcal{P}_0(1) \subset \mathcal{P}_0(0) = \mathcal{P}_0, \tag{5.5}$$

which yields the inequality:

$$Re\{f'(z)\} > 0 \quad (z \in \mathcal{U}). \tag{5.6}$$

Therefore, by using the same arguments as above, we find from (5.6) that $f \in \mathcal{H}^p$ for all $p < \infty$. Thus, in particular, $f \in \mathcal{H}^1$ and $D^1 f \in \mathcal{H}^1$.

Next, by comparing (4.18) and (4.19), we obtain

$$f'(z) = \frac{1}{z}\{D^1 f(z) - f(z)\}, \tag{5.7}$$

which readily yields

$$\int_0^{2\pi} |f'(re^{i\theta})| d\theta \leq \frac{1}{r}\{\int_0^{2\pi} |D^1 f(re^{i\theta})| d\theta + \int_0^{2\pi} |f(re^{i\theta})| d\theta\} \quad (r = |z|) \tag{5.8}$$

or equivalently,

$$M_1(r, f') \leq \frac{1}{r}\{M_1(r, D^1 f) + M_1(r, f)\}. \tag{5.9}$$

Proceeding to the limit as $r \to 1$, we find from this last inequality (5.9) that

$$\lim_{r \to 1}\{M_1(r, f')\} < \infty, \tag{5.10}$$

showing that $f' \in \mathcal{H}^1$. Thus, by applying another known result [2, p.42, Theorem 3.11], we conclude that f is continuous in

$$\bar{\mathcal{U}} = \mathcal{U} \cup \partial\mathcal{U} = \{z : |z| \leq 1\}.$$

Finally, since $\bar{\mathcal{U}}$ is compact, f is bounded in $\bar{\mathcal{U}}$. Hence f is a bounded analytic function in $\mathcal{U}$, which completes the proof of Theorem 4.

As an interesting consequence of Theorem 4 and Corollary 2, we have

COROLLARY 3. *If*

$$f \in \mathcal{P}_\alpha(\beta) \quad (0 \leq \alpha < 1; \beta \geq 1), \tag{5.11}$$

then f is a bounded univalent function in $\mathcal{U}$.

6. Applications Involving Generalized Hypergeometric Functions

Let $\rho_j (j = 1, \ldots, r)$ and $\sigma_j (j = 1, \ldots, s)$ be complex numbers with

$$\sigma_j \neq 0, -1, -2, \ldots \quad (j = 1, \ldots, s). \tag{6.1}$$

Then the generalized hypergeometric function ${}_rF_s(z)$ is defined by (cf., e.g., [12, p.333])

$$\begin{aligned} {}_rF_s(z) &\equiv {}_rF_s(\rho_1, \ldots, \rho_r; \sigma_1, \ldots, \sigma_s; z) \\ &= \sum_{n=0}^{\infty} \frac{(\rho_1)_n \cdots (\rho_r)_n}{(\sigma_1)_n \cdots (\sigma_s)_n} \frac{z^n}{n!} \quad (r \leq s+1), \end{aligned} \tag{6.2}$$

where $(\lambda)_n$ denotes the Pochhammer symbol defined by

$$(\lambda)_n = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n = 0) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (n \in \mathbf{N}). \end{cases} \tag{6.3}$$

We note that the ${}_rF_s(z)$ series in (6.2) converges absolutely for $|z| < \infty$ if $r < s+1$, and for $z \in \mathcal{U}$ if $r = s+1$.

Applying Theorem 3 to the generalized hypergeometric function defined by (6.2), we can derive an interesting (presumably new) property of this important class of functions involving the space $\mathcal{P}_\alpha(\beta)$. More generally, we shall prove

THEOREM 5. *Let the function*

$$z\,{}_{r+1}F_{s+1}(\rho_1, \ldots, \rho_r, 1 + \lambda^{-1}; \sigma_1, \ldots, \sigma_s, \lambda^{-1}; z) \quad (r \leq s+1; \lambda > 0)$$

be in the class $\mathcal{P}_\alpha(\beta)$. Then the function

$$z_rF_s(\rho_1,\ldots,\rho_r;\sigma_1,\ldots,\sigma_s;z)$$

is in the class $\mathcal{P}_\delta(\beta)$ for δ given by

$$\delta=\frac{\lambda+2\alpha}{\lambda+2}\quad(\lambda>0;0\le\alpha<1). \tag{6.4}$$

Proof. From (1.15), (6.2), and (6.3), we have

$$\begin{aligned}(6.5)\quad & 2^{-\beta}D^\beta(z_{r+1}F_{s+1}(\rho_1,\ldots,\rho_r),1+\lambda^{-1};\sigma_1,\ldots,\sigma_s,\lambda^{-1};z)) \\ &= z+\sum_{n=1}^{\infty}[\frac{n+1}{2}]^\beta\frac{(\rho_1)_n\cdots(\rho_r)_n}{(\sigma_1)_n\cdots(\sigma_s)_n}\frac{(1+\lambda^{-1}}{(\lambda^{-1})_n}\frac{z^{n+1}}{n!} \\ &= z+\sum_{n=1}^{\infty}[\frac{n+1}{2}]^\beta(\lambda n+1)\frac{(\rho_1)_n\cdots(\rho_r)_n}{(\sigma_1)_n\cdots(\sigma_s)_n}\frac{z^{n+1}}{n!} \\ &= (1-\lambda)w(z)+\lambda zw'(z),\end{aligned}$$

where, for convenience,

$$\begin{aligned}(6.6)\quad w(z) &= z+\sum_{n=1}^{\infty}[\frac{n+1}{2}]^\beta\frac{(\rho_1)_n\cdots(\rho_r)_n}{(\sigma_1)_n\cdots(\sigma_s)_n}\frac{z^{n+1}}{n!} \\ &= 2^{-\beta}D^\beta(z_rF_s(\rho_1,\ldots,\rho_r;\sigma_1,\ldots,\sigma_s;z)).\end{aligned}$$

Now, in view of Definition 3 and a known result [10, p.580, Theorem 2], the assertion of Theorem 5 follows immediately from (6.5) and (6.6).

A special case of Theorem 5 when $\lambda=\frac{1}{2}$ (so that $\delta=\mathcal{M}u$, where δ and $\mathcal{M}u$ are given by (6.4) and (4.20), respectively) can indeed be derived *directly* from Theorem 3.

Finally, by applying Theorem 1 and Corollary 1, we obtain

THEOREM 6. *Let the function*

$$z_rF_s(\rho_1,\ldots,\rho_r;\sigma_1,\ldots,\sigma_s;z)\quad(r\le s+1)$$

be in the class $\mathcal{P}_\alpha(\beta)$. Then the function

$$z_{r+p}F_{s+p}(\rho_1,\ldots,\rho_r,c_1+1,\ldots,c_p+1;\sigma_1,\ldots,\sigma_s,c_1+2,\ldots,c_p+2;z)$$

is also in the class $\mathcal{P}_\alpha(\beta)$ for $c_j>-1(j=1,\ldots,p)$.

The proof of Theorem 6 is much akin to that of Theorem 4 (and Corollary 3) of Owa and Srivastava [11, p.128]. The details may be omitted.

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