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ON A SUFFICIENT CONDITION FOR M–IDEAL PROPERTY OF X IN X^{**}

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0. Introduction

The notion of an M-ideal was introduced by Alfsen and Effros [1] and has been studied in various aspects [1–10, 12–14]. A weaker notion is a semi M-ideal.

If a closed subspace J of a Banach space X is a semi M-ideal, then J is a proximinal subspace of X, i.e. if $x \in X \setminus J$ then there exists $j \in J$ such that $\|x - j\| = \inf\{\|x - y\| : y \in J\}$. Such a $j \in J$ is called a best approximation of x in J. An M-ideal has a rather surprising approximation property. In fact, if J is an M-ideal in a Banach space X, then for each $x \in X \setminus J$, the set of all best approximations of x in J actually spans J [7].

Smith and Ward [14] proved that (i) M-ideals in a C^*-algebra are precisely the closed two sided ideals, (ii) M-ideals in a commutative Banach algebra with the identity are ideals and (iii) M-ideals in a Banach algebra with the identity are algebras.

Lima [10] proved that if $K(X)$, the space of all compact linear operators on a Banach space X, satisfies the 2-ball property in $L(X)$, the space of all continuous linear operators on X, then X satisfies the 2-ball property in the second dual space X^{**} and hence by a result of Saatkamp [13] X is an M-ideal in X^{**}.

In Theorem 2.2, we will prove that X is an M-ideal in X^{**} under a ball property which is weaker than the 2-ball property of $K(X)$.

1. Preliminaries

Let W and X be Banach spaces. $L(W, X)$ will denote the space of all bounded linear operators from W to X with the operator norm.

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$K(W,X)$(resp. $K_w(W,X)$) will denote the space of all compact(resp. weakly compact) operators from W to X with the operator norm.

In a Banach space X, $B(a,r)$ will denote the closed ball with the center a and the radius r. If $a=0$ and $r=1$, then we will simply write B_X for $B(0,1)$.

A closed subspace J of a Banach space X is called an L-summand if there is a projection P on X such that $PX = J$ and $\|x\| = \|Px\| + \|(I-P)x\|$ for every $x \in X$. A closed subspace J of X is called a semi L-summand if for every $x \in X$ there is a unique $y \in J$ such that $\|x-y\| = \inf\{\|x-z\| : z \in J\}$, and moreover this y satisfies $\|x\| = \|y\| + \|x-y\|$. A closed subspace J of X is M-ideal(resp. a semi M-ideal) if J^0, the annihilator of J in X^*, is an L-summand(resp. a semi L-summand) in X^*.

The following characterizations of an M-ideal and a semi M-ideal are due to Lima [8].

THEOREM 1.1 [8]. *For a closed subspace J of a Banach space X, the following statements are equivalent :*

(1) *J is an M-ideal in X.*
(2) *J satisfies the n-ball property for every $n \geq 3$. That is, if $\{B(a_i,r_i)\}_{i=1}^n$ is a family of closed balls in X such that*

$$\bigcap_{i=1}^{n} B(a_i,r_i) \neq 0 \quad \text{and} \quad J \cap B(a_i,r_i) \neq 0$$

for each i, then for every $\varepsilon > 0$

$$J \cap \bigcap_{i=1}^{n} B(a_i, r_i+\varepsilon) \neq 0,$$

(3) *J satisfies the 3-ball property.*

THEOREM 1.2 [8]. *For a closed subspace of a Banach space X, the following statements are equivalent :*

(1) *J is a semi M-ideal in X.*
(2) *J satisfies the 2-ball property.*

(3) *For any $x \in B_X$ and any $j \in J$ with $\|j\| = 1$, we have*

$$J \cap B(x + j, 1 + \varepsilon) \cap B(x - j, 1 + \varepsilon) \neq \emptyset$$

for all $\varepsilon > 0$.

2. Results

Recall that a linear operator Q from W to X is a quotient map if $Q(B_W)$ is norm dense in B_X.

LEMMA 2.1. *If $T : W \to X$ is a quotient map, then the second adjoint $T^{**} : W^{**} \to X^{**}$ of T is also a quotient map.*

Proof. First note that $T^* : X^* \to W^*$ is an isometry. In fact, for $x^* \in X^*$ we have

$$\begin{aligned}\|T^* x^*\| &= \sup\{|x^*(Tw)| : w \in B_W\} \\ &= \|x^*\|\end{aligned}$$

Thus T^* is an isometry.

If $x^{**} \in X^{**}$, then $x^{**} \circ (T^*)^{-1} \in (T^*(X^*))^*$. Let $w^{**} \in W^{**}$ be a norm preserving extension of $x^{**} \circ (T^*)^{-1}$ to W^*. Then $T^{**}(w^{**}) = w^{**} \circ T^* = (x^{**} \circ (T^*)^{-1}) \circ T^* = x^{**}$ and $\|x^{**}\| = \|w^{**}\|$.

THEOREM 2.2. *Suppose W and X are Banach spaces and suppose there exists a quotient map $Q : W \to X$. If $K_w(W, X) \cap B(T + S, 1 + \varepsilon) \cap B(T - S, 1 + \varepsilon) \neq \emptyset$ holds for any $T \in B_{L(W,X)}$, any norm one rank one operator $S : W \to X$ and any $\varepsilon > 0$, then X is an M-ideal in X^{**}.*

Proof. By a result of Saatkamp [13], X is an M-ideal in X^{**} if and only if X is a semi M-ideal in X^{**}. So it suffices to prove that X as a closed subspace of X^{**} has the property in Theorem 1.2 (3).

Let $x^{**} \in B_{X^{**}}$, $x_0 \in X$ with $\|x_0\| = 1$ and $\varepsilon > 0$. We will find $z \in X$ such that $\|x^{**} \pm x_0 - z\| < 1 + \varepsilon$.

Let $\|x^{**}\| = \alpha$. Since Q^{**} is also a quotient map, there exists $w^{**} \in W^{**}$ with $\|w^{**}\| = 1$ such that

$$\|\alpha Q^{**} w^{**} - x^{**}\| < \frac{\varepsilon}{3}$$

Choose $f \in W^*$ such that $1 = \|f\| \geq w^{**}(f) > 1 - \frac{\varepsilon}{3}$, and define a rank one operator $S : W \to X$ by

$$S(x) = f(x)x_0.$$

Then $\|S\| = 1$. By assumption, there exists a weakly compact operator $K : W \to X$ such that

$$\|\alpha Q \pm S - K\| < 1 + \frac{\varepsilon}{3}$$

and hence we have

$$\|\alpha Q^{**}w^{**} \pm S^{**}w^{**} - K^{**}w^{**}\| < 1 + \frac{\varepsilon}{3}.$$

Since $S^{**}w^{**} = w^{**}(f)x_0$, we get

$$\|x^{**} \pm x_0 - K^{**}w^{**}\| < 1 + \varepsilon.$$

Since K is weakly compact, $z = K^{**}w^{**}$ is in X and our proof is complete.

COROLLARY 2.3. *Let X be a Banach space. If either $K(X,X)$ or $K_w(X,X)$ has the 2-ball property, then X is an M-ideal in X^{**}.*

REMARK. For certain Banach spaces W and X, $K_w(W,X)$ is very large. If either W or X is reflexive, then every bounded linear operator from W to X is weakly compact. The same is true if X is weakly complete and $W = C(K)$, the space of continuous functions on a compact Hausdorff space K. In theses cases, the ball property in Theorem 2.2 is automatically satisfied. For any pair of Banach spaces W and X, the 2-ball property of $K(W,X)$ in $L(W,X)$ implies the ball property in Theorem 2.2.

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