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ISOMORPHIC ORE EXTENSIONS OF MONOMORPHISM TYPE

JANG-HO CHUN

1. Introduction

Throughout this paper, all rings have identity and all ring homomorphisms preserve the identity. Let A be a ring and α be a monomorphism on A. Let $A[x : \alpha]$ denote the set of all formal right polynomials in x with coefficients in A written on right of powers of x. Define addition in $A[x : \alpha]$ as usual and define a multiplication by assuming the distributive laws and the rule $ax = xa^{\alpha}$ for all $a \in A$. It is straightforward to check that $A[x : \alpha]$ is a ring and this ring is called an *Ore extension of monomorphism type (or skew polynomial ring).* Given two Ore extensions (of the same type) $A[x : \alpha]$ and $B[y : \beta]$, we denote the fact that $A[x : \alpha]$ and $B[y : \beta]$ are isomorphic via the ring isomorphism σ by writing $A[x; \alpha] \overset{\sigma}{\cong} B[y; \beta]$. Suppose α is a monomorphism of A and β is a monomorphism of B. We say that A is *Ore invariant of monomorphism type* if whenever $A[x; \alpha] \overset{\sigma}{\cong} B[y; \beta]$, then we have $A \cong B$. If, furthermore, the isomorphism σ carries A onto B then A is said to be *strongly Ore invariant of monomorphism type.*

The purpose of this paper is to decide the rings which are (strongly) Ore invariant of monomorphism type. Several authors have considered the invariance of polynomial ring problem which is a special case of Ore invariance of monomorphism type (e.g., [2],[4],[5]). In particular, in [4], it is shown that every right or left Artinian ring is polynomial invariant. In [2], the invariance of polynomial ring was extended to n variables and they obtained that a finite direct sum of local rings is n-invariant for each positive integer n. While in [6], an example of a noninvariant commutative Noetherian domain was presented. Invariance of Ore extensions of derivation type was considered in [1]. It was proved

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that any simple Artinian ring of characteristic zero or any regular self-injective P.I. ring with no Z-torsion is Ore invariant of derivation type. In [3], the author observed the Ore invariance of automorphism type. In this paper, we try to extend the results in [3] for the monomorphism case. We obtain that every semisimple Artinian ring is Ore invariant of monomorphism type. The proof of this heavily depends on the structure of the Ore extension of monomorphism type over a semisimple Artinian ring which has been carried out by A. V. Jategaonkar in [9] and [10]. In section 2, we will introduce some of his results which will be needed in the proof of our theorems.

2. Notations and Preliminaries

In general, the structure of $A[x:\alpha]$ is very complicate, but it is well observed by A. V. Jategaonkar in [9] and [10]. Hence we introduce his results in this section using the same notations.

Let $K_1 \subseteq \cdots \subseteq K_m$ be a chain of rings and $\varphi : K_m \to K_1$ be a monomorphism. Let $D_i = K_i[z:\varphi]$, $i = 1,\ldots,m$. Let P be the subring of $M_n(D_m)$, consisting of all those matrices (d_{ij}) which satisfy the following conditions: (1) $d_{ij} \in D_j$ for all i,j (2) $d_{ij} \in zD_j$ if $i > j$. It is straightforward to check that P is a subring of $M_n(D_m)$. We shall denote P by $\{K_i, m, \varphi, z\}$. The following theorem says that the Ore extention of monomorphism type for self-basic semisimple ring is isomorphic to $\{K_i, m, \varphi, z\}$.

THEOREM. A ([9]). *Let R be a semisimple Artinian ring, $\{f_1,\ldots,f_m\}$ be the set of all distinct central primitive idempotents in R. Assume that $\alpha : R \to R$ is a monomorphism such that $\alpha(f_i) = f_{\pi(i)}$ where π is the cycle on $I_m = \{1,2,\ldots,m\}$ defined by $\pi(m) = 1$ and $\pi(i) = i+1$ otherwise. Then*

(1) *There exists a chain of division rings $K_1 \subseteq \cdots \subseteq K_m$, a monomorphism $\varphi : K_m \to K_1$ and a positive integer n such that*

$$R[x:\alpha] \cong M_n(\{K_i, m, \varphi, z\}).$$

α is an automorphism if and only if $K_1 = \cdots = K_m$ and φ is an automorphism.

(2) *$R[x:\alpha]$ is a right and left hereditary, right and left Noetherian prime ring of right rank m and a right order in a simple ring.*

(3) *If M is a finitely generated projective right $R[x:\alpha]$-module, then M can be uniquely expressed as*

$$M \cong \bigoplus_{i=1}^{m} (f_i R[x:\alpha])^{n_i}$$

where the n_i are nonnegative integers and $R[x:\alpha]$ has precisely m distinct isomorphism classes of uniform right projective modules; further, $\{f_i R : i \in I_m\}$ is a representative set of those isomorphism classes.

To prove the Ore invariance of monomorphism type for simple Artinian rings, we need some properties of matrix rings which have been established in [11] and [12]. Let $\rho : S \to R$ be an isomorphism of rings and let A, B be left S- and R-modules, respectively. A map $\omega : A \to B$ is called a ***ρ-semi-linear homomorphism*** if ω is a homomorphism of additive Abelian groups $(A,+)$ to $(B,+)$ and $\omega(sa) = \rho(s)\omega(a)$ for all $s \in S$ and $a \in A$. Given a ρ-semi-linear isomorphism $\omega : A \to B$, the map

$$\sigma : \mathrm{End}_S(A) \to \mathrm{End}_R(B)$$

defined by $\alpha^\sigma = \omega\alpha\omega^{-1}$, $\alpha \in \mathrm{End}_S(A)$ is called the *isomorphism induced by ω*.

THEOREM. B ([11], [12]). *For a fixed positive integer n, the following conditions on a ring R are equivalent.*

(1) *If P is a left R-module with $P^{(n)} \cong R^{(n)}$ then $P \cong R$ as R-modules, where $P^{(n)}$ means a direct sum of n copies of P.*

(2) *If S is a ring and*

$$\sigma : M_n(S) \to M_n(R)$$

is an isomorphism, then there exists an isomorphism $\rho : S \to R$ and a ρ-semi linear isomorphism $\omega : S^{(n)} \to R^{(n)}$ such that σ is induced by ω.

(3) *Every endomorphism of $M_n(R)$ is of the form $x \mapsto uf(x)u^{-1}$ where f is an extended endomorphism of R and u is an element in $M_n(R)$.*

A ring which satisfies the above equivalent conditions is called a *PF_n-ring*. We know that every pri-domain satisfies the condition (1).

3. Ore invariance of simple Artinian Rings

We first consider $A[x : \alpha]$ where A is a division ring. It is well-known ([7]) that $A[x : \alpha]$ has a right Euclidean algorithm so that it is a pri-domain. The proof presented here is a modification of the proof of ordinary polynomial rings or Ore extensions of derivation type.

PROPOSITION 3.1. *Let A be a division ring. Then A is strongly Ore invariant of monomorphism type, i.e., whenever $A[x;\alpha] \overset{\sigma}{\cong} B[y;\beta]$, we have $\sigma(A) = B$.*

Proof. For a ring R, let R^* denote the multiplicative group of units. Since $A[x : \alpha]$ is a domain and noncommutative polynomial ring, we have $(A[x : \alpha])^* = A^*$. Therefore

$$\sigma(A^*) = \sigma(A[x : \alpha])^* = (B[y : \beta])^* = B^*.$$

Hence $\sigma(A) \subseteq B$. Next we will show that $1, \sigma(x), \sigma(x)^2, \ldots$ are linearly independent over B. Suppose that

$$a_0 + \sigma(x)a_1 + \cdots + \sigma(x)^n a_n = 0$$

for some $a_i \in B$. Let $\sigma(x) = b_0 + yb_1 + \cdots + y^m b_m$, then $m \geq 1$ and $b_m \neq 0$. The highest term of the above equation is $y^{nm} b_m^{\beta^{(n-1)m}} b_m^{\beta^{(n-2)m}} \cdots b_m a_n$. Therefore $b_m^{\beta^{(n-1)m}} \cdots b_m a_n = 0$. Since B is a domain and $b_m^{\beta^{im}} \neq 0$ for all $i = 0, 1, \ldots, n-1$, $a_n = 0$. Continuing this way, we get $a_0 = a_1 = \cdots = a_n = 0$. Let b be arbitrary element in B. Then

$$b = \sigma(a_0 + xa_1 + \cdots + x^n a_n).$$

Since $1, \sigma(x), \ldots, \sigma(x)^n$ are linearly independent over B and $\sigma(a_i) \in B$, $\sigma(a_0) = b$. Hence $\sigma(A) = B$.

THEOREM 3.2. *If A is a simple Artinian ring and $A[x;\alpha] \overset{\sigma}{\cong} B[y;\beta]$, then we have $A \cong B$.*

Proof. In general, $A[x : \alpha]$ is not necessary right Noetherian even if A is right Artinian. For example, see [10]. But if A is a simple Artinian ring, by [12], $A[x : \alpha]$ is right Noetherian. So is $B[y : \beta]$. Since B is naturally isomorphic to $B[y:\beta]/(y)$, B is also right Noetherian. Next we will show that B is right Artinian. We know that $A \cong M_n(D)$ for some division ring D and $M_n(D)[x:\alpha] \cong M_n(D[ux:\rho])$ for some $u \in M_n(D)$. Therefore we have

$$B \cong \frac{B[y:\beta]}{(y)} \cong \frac{A[x:\alpha]}{(\sigma^{-1}(y))} \cong \frac{M_n(D)[x:\alpha]}{(\sigma^{-1}(y))} \cong \frac{M_n(D[ux:\rho]}{I}$$

where I is an ideal of $M_n(D[ux:\rho])$. Since $D[ux:\rho]$ is a pri-domain, I is of the form $(a_0+xa_1+\cdots+x^n)D[ux:\rho]$ where $a_i \in D$. $D[ux:\rho]$ satisfies the right division algorithm, and hence by the usual method, $D[ux:\rho]/I$ is right Artinian. This implies that B is right Artinian. By [10], $A[x:\alpha]$ is a right order of simple Artinian ring and hence B is semisimple and β induces a cyclic permutation on the set of central primitive idempotents on B. Suppose that $\{f_1,\ldots,f_m\}$ is the set of all distinct central primitive idempotents in B. Then by [9], $B[y:\beta]$ is right hereditary prime ring of right rank m. Since $A[x:\alpha] \cong M_n(D[ux:\rho])$ and pri-ring, $n=1$. Therefore B is a simple Artinian ring, i.e., $B \cong M_l(F)$ for some division ring F. And then we get the following isomorphism

$$A[x:\alpha] \cong M_n(D[ux:\rho]) \cong M_l(F[vy:\mu]) \cong B[y:\beta].$$

Since $D[ux:\rho]$ is pri-domain, by theorem B, $D[ux:\rho] \cong F[vy:\mu]$. By theorem 3.1, $D \cong F$ and hence $A \cong B$.

4. Ore invariance of semisimple Artinian ring

First we consider that A is a self-basic semisimple ring, i.e., A is a finite sum of division rings.

LEMMA 4.1. *Let A be a self-basic semisimple ring and $\{f_1,\ldots,f_m\}$ be the set of all distinct primitive idempotents of A. Let π be the cycle on I_m defined by $\pi(m)=1$ and $\pi(i)=i+1$ for $i \neq m$. Assume that α is a monomorphism on A such that $\alpha(f_i)=f_{\pi(i)}$ and α is not an automorphism. Then $A[x:\alpha] \cong \{K_i,m,\varphi,z\}$ and every nonzero ideal of $A[x:\alpha]$ is of the form $(z^{\epsilon_{ij}}D_j)$ where $\epsilon_{ij} \geq 0$.*

Proof. See [9].

LEMMA 4.2. *Let A be a semisimple Artinian ring and α a cycle permutation on the set of all central primitive idempotents $\{e_1, \ldots, e_m\}$. If $A[x; \alpha] \overset{\sigma}{\cong} B[y; \beta]$, then B is also semisimple Artinian.*

Proof. The proof is divided into two parts, one is α is an automorphism and the other is α is not an automorphism.
Case I. α is an automorphism. Define $f(a)$ to be the constant term of $\sigma(a)$ for all $a \in A$. Then clearly f is a homomorphism from A into B. Since any element in (y) cannot be an idempotent, for any nonzero idempotent $e \in A$, $f(e)$ is a nonzero idempotent in B. Since (y) is a proper ideal in $B[y : \beta]$, $\sigma^{-1}((y)) \cap A$ is also a proper ideal in A. If $\sigma^{-1}((y)) \cap A \neq 0$, then $\sigma^{-1}((y)) \cap A$ contains at least one e_i, because every nonzero ideal in A is of the form $\sum_{j \in J} e_j A$ where $\emptyset \neq J \subseteq I_n$. This is a contradiction to the above observation. Hence $\sigma^{-1}((y)) \cap A = 0$. Therefore f is a one-to-one homomorphism. Since $\sum_{i=1}^{n} e_i = 1$, we also have $\sum_{i=1}^{n} f(e_i) = 1$ in B. Therefore $b = \sum_{i=1}^{n} bf(e_i)$ for all $b \in B$. Let

$$\sigma(x) = b_0 + yb_1 + \cdots + y^t b_t.$$

Then B is generated by $f(A)$ and b_0 as a ring, because for any element $b \in B$, there exists

$$a_0 + xa_1 + \cdots + x^l a_l \in A[x : \alpha]$$

such that

$$\sigma(a_0) + \sigma(x)\sigma(a_1) + \cdots + \sigma(x)^l \sigma(a_l) = b,$$

i.e.,

$$b = f(a_0) + b_0 f(a_1) + \cdots + b_0^l f(a_l).$$

Specially,

$$b_0 = f(c_0) + b_0 f(c_1) + \cdots + b_0^p f(c_p)$$

for some $c_i \in A$. Then

$$b_0 f(e_i) = f(c_0 e_i) + b_0 f(c_1 e_i) + \cdots + b_0^p f(c_p e_i).$$

If $f(e_i) b_0 = 0$ for all i, then $b_0 = 0$. This means $\sigma(x) \in (y)$. If

$$\sigma(d_0 + xd_1 + \cdots + x^j d_j) = y,$$

then $\sigma(d_0) \in (y)$. But $\sigma^{-1}((y)) \cap A = 0$ implies that $d_0 = 0$. Therefore $\sigma((x)) = (y)$. Then σ induces an isomorphism

$$\overline{\sigma} : \frac{A[x:\alpha]}{(x)} \longrightarrow \frac{B[y:\beta]}{(y)}.$$

This says that $A \cong B$. Hence we may assume that $b_0 f(e_i) \neq 0$ for some i. If $b_0 f(e_i) \neq 0$, then for some k, $f(c_k e_i) \neq 0$. Choose k_i maximal among $f(c_k e_i) \neq 0$. Since $f(c_{k_i}) f(e_i) \in f(A)f(e_i)$ and $f(A)f(e_i)$ is a simple ring with identity $f(e_i)$, there exists $s_{ij},\ t_{ij} \in A$ such that

$$\sum_j f(s_{ij}e_i) f(c_{k_i}e_i) f(t_{ij}e_i) = f(e_i).$$

The relation $ax = xa^\alpha$ implies that $\sigma(a)\sigma(x) = \sigma(x)\sigma(a^\alpha)$. Hence $f(a)b_0 = b_0 f(a^\alpha)$. In general, $f(a)b_0^t = b_0^t f(a^{\alpha^t})$ for all nonnegative integers t. Since α is an automorphism, we may choose $f(t'_{ij}e_i)$ such that $f(t'_{ij}e_i)b_0^{k_i} = b_0^{k_i} f(t_{ij}e_i)$. Then we have

$$\begin{aligned} f(s_{ij}e_i)b_0 f(e_i) f(t'_{ij}e_i) &= f(s_{ij}e_i) f(c_0 e_i) f(t'_{ij}e_i) \\ &\quad + \cdots + b_0^{k_i} f(s_{ij}e_i) f(c_{k_i}e_i) f(t_{ij}e_i) \end{aligned}$$

for each j. Therefore $b_0^{k_i} f(e_i) = \sum_{p=0}^{k_i-1} b_0^p a_{ip}$ where $a_{ip} \in f(A)$. Choose $k = max\{k_i\}$. Then

$$b_0^k = \sum_{i=1}^{n} b_0^k f(e_i) = \sum_{p=0}^{k_i-1} b_0^p \Big(\sum_i a_{ip}\Big)$$

where $a_{ip} \in f(A)$. This means that B is a finitely generated $f(A)$-module. Hence B is right Artinian.

Case II. α is not an automorphism. Clearly $B \cong A[x:\alpha]/(\sigma^{-1}(y))$. By Lemma 4.1, $(\sigma^{-1}(y))$ is of the form $(z^{\epsilon_{ij}} D_j)$ where $\epsilon_{ij} \geq 0$ and by [9], $A[x:\alpha]/(\sigma^{-1}(y))$ is a generalized uniserial ring. Therefore in both cases B is right Artinian. By [10], $B[y:\beta]$ is right order in a simple Artinian ring and B is a semisimple Artinian.

LEMMA 4.3. *$R \equiv \{K_i, n, \varphi, z\}$ is PF_p-ring for all positive integer p.*

Proof. Suppose that $P^{(p)} \cong R^{(p)}$ where P is a left R-module. Since R is right and left hereditary and P is a finitely generated projective R-module, by [9], $P \cong \bigoplus_i (e_i R)^{n_i}$. Then we have

$$P^{(p)} \cong \bigoplus_i (e_i R)^{pn_i}.$$

And since

$$\bigoplus_i (e_i R)^p \cong R^{(p)}$$

and $P^{(p)} \cong R^{(p)}$, by the uniqueness of the expression of finitely generated projective R-modules $pn_i = p$, i.e., $n_i = 1$ for all i. Hence

$$P \cong \bigoplus_{i=1}^{n} e_i R \cong R$$

as left R-modules.

THEOREM 4.4. *Let A be a semisimple Artinian ring and α a cycle permutation on the set of central idempotents $\{e_1, \ldots, e_n\}$. Suppose that $A[x;\alpha] \overset{\sigma}{\cong} B[y;\beta]$, then $A \cong B$.*

Proof. By Lemma 4.3, B is also a semisimple Artinian ring. Hence $A[x:\alpha] \cong M_p(\{K_i, n, \varphi, z\}) \cong M_q(\{L_i, n, \psi, w\}) \cong B[y:\beta]$ for some positive integers p and q. Since $\{K_i, n, \varphi, z\}$ is prime Noetherian, it has a total quotient ring $Q(\{K_i, n, \varphi, z\})$ and by [9], the set P consisting of diagonal matrices (d_{ii}) such that $degd_{ii} = degd_{jj}$ is the exhaust divisor set of $\{K_i, n, \varphi, z\}$. Therefore an easy calculation shows that $Q(\{K_i, n, \varphi, z\}) \cong M_n(Q(K_n[z:\varphi]))$. Therefore

$$Q(A[x:\alpha]) \cong M_{np}(Q(K_n[z:\varphi])) \cong M_{nq}(Q(L_n[w:\psi])) \cong Q(B[y:\beta]).$$

Since $Q(K[z:\varphi])$ is a division ring, $np = nq$, i.e., $p = q$. Hence $M_p(\{K_i, n, \varphi, z\}) \cong M_p(\{L_i, n, \psi, w\})$. By Lemma 4.3, $\{K_i, n, \varphi, z\}$ is a PF_n-ring and hence $R = \{K_i, n, \varphi, z\} \overset{\eta}{\cong} \{L_i, n, \psi, w\} = S$ for some isomorphism η. Let f_i and h_i be the standard orthogonal idempotents in R and S, respectively. Then $\{\eta(f_i)\}_{i=1}^{n}$ is also a set of orthogonal idempotents in S. Then

$$S = \{L_i, n, \psi, w\} \cong \bigoplus_{i=1}^{n} h_i S \cong \bigoplus_{i=1}^{n} \eta(f_i) S$$

as S-modules. Since $f_i R$ is uniform projective for each i, $\eta(f_i R) = \eta(f_i)S$ are also uniform projectives. Since $\{f_i R\}_{i=1}^{n}$ is a representative set of uniform projectives, $\{\eta(f_i)S\}_{i=1}^{n}$ are mutually nonisomorphic uniform projective S-modules. Therefore by suitable reordering, $\eta(f_i)S \cong h_i S$, for all $i \in I_n$. This implies that $\mathrm{End}_S(\eta(f_i)S) \cong \mathrm{End}_S(h_i S)$. Clearly the isomorphism η induces an isomorphism $\overline{\eta} : f_i R f_i \to \eta(f_i) S \eta(f_i)$. Hence we have an isomorphism

$$\begin{aligned} K_i[z : \varphi] \cong f_i R f_i \overset{\overline{\eta}}{\cong} \eta(f_i) S \eta(f_i) &\cong \mathrm{End}_S(\eta(f_i)S) \\ &\cong \mathrm{End}_S(h_i S) \cong h_i S h_i \cong L_i[\omega : \psi]. \end{aligned}$$

Since K_i is a division ring, $K_i \cong L_i$. Therefore we get

$$A \cong \bigoplus_i M_p(K_i) \cong \bigoplus_i M_p(L_i) \cong B,$$

completing the proof.

As a corollary we obtain one of the principal results of this paper.

THEOREM 4.5. *If A is semisimple Artinian and $A[x;\alpha] \overset{\sigma}{\cong} B[y;\beta]$, then $A \cong B$.*

Proof. Let $\{f_1, \cdots, f_m\}$ be the set of all distinct central primitive idempotents of A. Since α is a monomorphism, by [9], there exists a unique permutation π on I_m, such that $f_i^{\alpha} = f_{\pi(i)}$. Let $\pi = \pi_1 \cdots \pi_t$ be a decomposition of π into mutually disjoint cycles (we write 1-cycles also). Put

$$g_l = \sum_{i \in \pi_l} f_i, \quad l = 1, \ldots, t,$$

where $i \in \pi_l$ means i occurs in the cycle notation of π_l. Then for each $l \in I_t$, $\alpha(g_l) = g_l$. Also α induces an isomorphism

$$\alpha_l : g_l A \to g_l A$$

and $\{f_i : i \in \pi_l\}$ is the set of all distinct primitive idempotents in $g_l A$. Then

$$A[x : \alpha] \cong \bigoplus_{l=1}^{t} (g_l A)[g_l x : \alpha_l].$$

For each l, $g_l A$ is semisimple Artinian. Since $\{g_l\}$ is a set of central orthogonal idempotents in $A[x : \alpha]$ and $\sum_{l=1}^{t} g_l = 1$, so is $\{\sigma(g_l)\}$. Let $f(a)$ be the constant term of $\sigma(a)$ for all $a \in A$. Then by the same reasoning as in Lemma 4.2, $\{f(g_l)\}$ is a set of central orthogonal idempotents in B and $\sum_{l=1}^{t} f(g_l) = 1$. Therefore

$$B \cong \bigoplus_{l=1}^{t} f(g_l)B.$$

Since $\sigma(g_l)y = y\sigma(g_l)$, we have $f(g_l) = f(g_l)^{\beta}$. Therefore

$$\begin{aligned} A[x : \alpha] &\cong (\bigoplus_{l=1}^{t} g_l A)[x : \alpha] \cong \bigoplus_{l=1}^{t} (g_l A)[g_l x : \alpha_l] \\ &\cong \bigoplus_{l=1}^{t} (f(g_l)B)[f(g_l)y : \beta_l] \cong B[y : \beta]. \end{aligned}$$

Therefore

$$(f(g_l)B)[f(g_l)y : \beta_l] \cong (g_l A)[g_l x : \alpha_l].$$

By the previous theorem, $f(g_l)B \cong g_l A$ for each l. Hence $A \cong B$, as desired.

References

1. E. P. Armendariz, H. K. Koo and J. K. Park, *Isomorphic Ore extensions*, Comm. Algebra **15** (1987), 2633–2652.

2. J. W. Brewer and E. A. Rutter, *Isomorphic polynomial ring*, Arch. Math. **23** (1972), 484–488.
3. J. H. Chun, *Isomorphic Ore extensions of automorphism type*, Ph.D. Thesis, U. of Texas, Austin, 1988.
4. D. B. Coleman and E. E. Enochs, *Isomorphic polynomial rings*, Proc. Amer. Math. Soc. **27** (1972), 247–252.
5. P. Eakin and K. K. Kubota, *A note on the uniqueness of rings of coefficients in polynomial rings*, Proc. Amer. Math. Soc. **32** (1972), 333–341.
6. M. Hochster, *Nonuniqueness of coefficient rings in a polynomial rings* Proc. Amer. Math. Soc. **34** (1972), 81–82.
7. N. Jacobson, *The theory of rings*, Amer. Math. Soc. Math. Surveys, vol. 2, Providence 1943.
8. ———, *The structure of rings*, Amer. Math. Soc. Colloq. Publ., vol. 37, Providence 1964.
9. A. V. Jategaonkar, *Skew polynomial rings over semisimple rings*, J. Algebra **19** (1971), 315–328.
10. ———, *Skew polynomial rings over orders in Artinian rings*, J. Algebra **21** (1972), 51–59.
11. ———, *Left principal ideal rings*, Lecture notes in Math., vol. 123, Springer-Verlag, Berlin, 1970.
12. G. Renault, *Anneaux de polynomes de Ore sur des anneaux Artiniens*, Comm. Algebra **7** (1979), 753–761.

Department of Mathematics
Yeungnam University
Gyongsan 712-749, Korea