Comm. Korean Math. Soc. 8 (1993), No. 3, pp. 799–809

SOME REMARKS ON THE AUTOMATA-HOMOMORPHISMS

CHIN-HONG PARK

DEFINITION. (1) An *automaton*, $A = (M, X, \delta)$, is a triple where M is a nonempty set (the set of states), X is a nonempty set (the set of inputs), δ is a function (called the state transition function) mapping $M \times X$ into M. Also, we shall assume the useful property that $\delta(m, st) = \delta(\delta(m, s), t)$ for all $s, t \in X$ and $m \in M$.

NOTE. (i) An automaton A means a triple (M, X, δ) and M does not mean an automaton. But the attribute "automaton" will be sometimes used for M. (ii) Let X^* be the free monoid generated by X. Then $\delta^* : M \times X^* \to M$ is the map defined as follows: For all $m \in M$ and $a \in X^*$, $\delta^*(m, a) = m$ if $a = e$ (empty string) and $\delta^*(m, a) = \delta(\delta^*(m, b), t)$ if $a = bt$ and $t \in X$.

NOTATION. For convenience we will denote $\delta(m, t)$ as mt if $t \in X$ and $\delta^*(m, a)$ as ma if $a \in X^*$, i.e., $\delta(m, t) = mt$ and $\delta^*(m, a) = ma$.

NOTE. (i) $\delta^*(m, t) = \delta(m, t)$ for all $m \in M$ and $t \in X$. (ii) $\delta^*(m, ab) = \delta^*(\delta^*(m, a), b)$ for all $m \in M$ and $a, b \in X^*$, i.e., $m(ab) = (ma)b$.

(2) Let $A = (M, X, \delta_A)$ and $B = (N, Y, \delta_B)$ be automata. An *automata-homomorphism* (or a *generalized XY-homomorphism*) of A into B is a pair (f, α) of mappings $f : M \to N$ and $\alpha : X \to Y$ such that $f(ma) = f(m)\alpha(a)$ for all $m \in M$ and $a \in X$.

NOTATION. We denote (f, α) as f^α, i.e., $f^\alpha = (f, \alpha)$.

(3) Let $A = (M, X, \delta_A)$ and $B = (N, X, \delta_B)$ be automata. Let $S = X^+ = X^* - \{e\}$. Let $f : M \to N$, $\alpha : X \to X$ and $\alpha^* : S \to S$ (or $X^* \to X^*$) be maps. Then $f^\alpha : A \to B$ is an *αX-homomorphism* (or *automata-homomorphism* or a *generalized X-homomorphism with respect to α*) if $f(ma) = f(m)\alpha(a)$ for all $m \in M$ and $a \in X$. Also, f^{α^*} is an *$\alpha^* S$-homomorphism* (resp. *$\alpha^* X^*$-homomorphism*) if $f(ma) =$

Received January 20, 1993.

$f(m)\alpha^*(a)$ for all $m \in M$ and $a \in S$ (resp. $f(ma) = f(m)\alpha^*(a)$ for all $m \in M$ and $a \in X^*$). f^α is an *αX-endomorphism* if $A = B$ and it is an αX-homomorphism. f^α is an *αX-isomorphism* if it is an αX-homomorphism and f, α are bijective. f^α is an *αX-automorphism* if it is an αX-isomorphism and $A = B$. Similarly, we can define $\alpha^* S$- and *$\alpha^* X^*$-endomorphismS*, $\alpha^* S$-and *$\alpha^* X^*$-isomorphisms* and $\alpha^* S$-and *$\alpha^* X^*$-automorphismS*. $f : M \to N$ (or $A \to B$) is an *X-homomorphism* (resp. *S-homomorphism*, *X^*-homomorphism*) if $f(ma) = f(m)a$ for all $m \in M$ and $a \in X$ (resp. for all $m \in M$ and $a \in S$, for all $m \in M$ and $a \in X^*$). Let f^α and g^β be an αX-homomorphism and an βX-homomorphism respectively. Then we define $f^\alpha = g^\beta$ by letting $f = g$ and $\alpha = \beta$

NOTATION. (i) We denote f^{id} as f where $id : X \to X$ (or $S \to S$ or $X^* \to X^*$) is the identity map.

(ii) $\mathrm{END}_X(A) = \mathrm{END}_X(M) = \{f^\alpha \mid f^\alpha$ is an αX-endomorphism with a map $\alpha : X \to X\}$.

$$\mathrm{AUT}_X(A) = \mathrm{AUT}_X(M) = \{f^\alpha \mid f^\alpha \text{ is an } \alpha X\text{-automorphism with a map } \alpha : X \to X\}.$$

$$\mathrm{End}_X(A) = \mathrm{End}_X(M) = \{f \mid f : M \to M \text{ is an } X\text{-endomorphism}\}.$$

$$\mathrm{Aut}_X(A) = \mathrm{Aut}_X(M) = \{f \mid f : M \to M \text{ is an } X\text{-automorphism}\}.$$

PROPOSITION 1. *Let $A = (M, X, \delta)$ be an automaton. For any $f^\alpha, g^\beta \in \mathrm{END}_X(A)$, we define $f^\alpha g^\beta = (fg)^{\alpha\beta}$. Then the following statements hold:* (1) *$\mathrm{END}_X(A)$ is a monoid and $\mathrm{End}_X(A)$ is a submonoid of $\mathrm{END}_X(A)$.*

(2) *$\mathrm{Aut}_X(A)$ and $\mathrm{AUT}_X(A)$ are groups where the product of maps means the composition of maps.*

LEMMA 2. *Let $A = (M, X, \delta)$ be an automaton. For any $f^\alpha, g^\beta, h^\gamma \in \mathrm{END}_X(A)$, we define two relations and operations on $\mathrm{END}_X(A)$ as follows:*

$$(f^\alpha, g^\beta) \in \sigma_E \iff f = g$$

$$(f^\alpha, g^\beta) \in \tau_E \iff \alpha = \beta$$

$$(f^\alpha, g^\beta)h^\gamma = (f^\alpha h^\gamma, g^\beta h^\gamma) \text{ and } h^\gamma(f^\alpha, g^\beta) = (h^\gamma f^\alpha, h^\gamma g^\beta).$$

Then σ_E and τ_E are congruences relations on $\mathrm{END}_X(A)$.

Proof. We will show that τ_E is a congruence relation on $\mathrm{END}_X(A)$. It is easy to show that τ_E is an equivalence relation. To show τ_E is a congruence relation, let $(f^\alpha, g^\alpha) \in \tau_E$. For any $h^\beta \in \mathrm{END}_X(A)$, $(f^\alpha, g^\alpha)h^\beta = (f^\alpha h^\beta, g^\alpha h^\beta) = ((fh)^{\alpha\beta}, (gh)^{\alpha\beta}) \in \tau_E$ and $h^\beta(f^\alpha, g^\alpha) = (h^\beta f^\alpha, h^\beta g^\alpha) = ((hf)^{\beta\alpha}, (hg)^{\beta\alpha}) \in \tau_E$. Similarly, it is easy to show that σ_E is a congruence relation.

NOTE. (1) $\mathrm{AUT}_X(A) \leq \mathrm{END}_X(A)$ and σ_A and σ_E are relations on $\mathrm{AUT}_X(A)$ and $\mathrm{END}_X(A)$ resp. (2) Similarly, for any $f^\alpha, g^\beta \in \mathrm{AUT}_X(A)$ we can define two congruence relations on $\mathrm{AUT}_X(A)$ as follows:

$$(f^\alpha, g^\beta) \in \sigma_A \iff f = g$$
$$(f^\alpha, g^\beta) \in \tau_A \iff \alpha = \beta.$$

Then (1) σ_A and τ_A are congruence relations on $\mathrm{AUT}_X(A)$.
(2) $\sigma_A \leq \sigma_E$, $\tau_A \leq \tau_E$ and $\mathrm{AUT}_X(A)/\tau_A = \mathrm{AUT}_X(A)/\mathrm{Aut}_X(A)$.

DEFINITION. Let $A = (M, X, \delta)$ be an automaton. Let $S = X^* - \{e\}$ and $a \in S$. (1) $T_a : M \to M$ is called a *right translation* if $T_a(m) = ma$ for all $m \in M$. (2) We define a congruence $\mu_M \subset S \times S$ on S through $(a, b) \in \mu_M \iff T_a = T_b$ for $a, b \in S$. (3) A (or M) is *cyclic* iff $M = mS$ for some $m \in M$. Also, m is called a *generator*. (4) A (or M) is *abelian* iff $m(st) = m(ts)$ for all $m \in M$ and $s, t \in S$. (5) A (or M) is *strongly connected* iff every element of M is a generator. (6) A (or M) is *perfect* iff A is strongly connected and abelian (see Fleck [4]).

PROPOSITION 3. *Let $A = (M, X, \delta)$ be an automaton. Then the following conditions are equivalent:*
(1) *$\mu_M = O$ on X where O is the identity relation.*
(2) *For all $a, b \in X$, $T_a = T_b \Longrightarrow a = b$.*
(3) *$\sigma_A = O$ on* $\mathrm{AUT}_X(A)$.
(4) *$\sigma_E = O$ on $\mathrm{END}_X(A)$ if A is perfect.*

Proof. (1) $\iff$ (2): Trivial. (2) $\Longrightarrow$ (3): Let $(f^\alpha, f^\beta) \in \sigma_A$. Since $f^\alpha, f^\beta \in \mathrm{AUT}_X(A)$, $f(ma) = f(m)\alpha(a) = f(m)\beta(a)$ for all $m \in M$ and $a \in X$. This means $T_{\alpha(a)}(f(m)) = T_{\beta(a)}(f(m))$. Since f is bijective, $T_{\alpha(a)}(m) = T_{\beta(a)}(m)$ for all $m \in M$. So, we have $T_{\alpha(a)} = T_{\beta(a)}$. By assumption, $\alpha(a) = \beta(a)$ for all $\alpha \in X$. Hence $\alpha = \beta$. i.e., $\sigma_A = O$. (3) $\Longrightarrow$ (2): We define a map $\alpha : X \to X$ given by $\alpha(a) = b$, $\alpha(b) = a$ and $\alpha(t) = t$ for all $t \in X - \{a, b\}$. Then α is bijective with $\alpha(\alpha(a)) = a$ and $\alpha(\alpha(b)) = b$. Moreover, $I^\alpha \in \mathrm{AUT}_X(A)$ (it is easy to show this, using $T_a = T_b$) and $(I^\alpha, I^{id}) \in \sigma_A$ where $id : X \to X$ is the identity map. Since $\sigma_A = O$, $I^\alpha = I^{id}$. Hence $\alpha = id$. This means that $a = b$. (2) $\Longrightarrow$ (4): Let $(f^\alpha, f^\beta) \in \sigma_E$. Since $f^\alpha, f^\beta \in \mathrm{END}_X(A)$, $f(ma) = f(m)\alpha(a) = f(m)\beta(a)$ for all $m \in M$ and $a \in X$. This implies $T_{\alpha(a)}(f(m)) = T_{\beta(a)}(f(m))$. Since M is perfect, from Lemma 1 of Park [1] and $T_S = \mathrm{End}_S(M)$ we have $T_{\alpha(a)} = T_{\beta(a)}$ where $T_S = \{T_a : a \in S\}$. By assumption, $\alpha(a) = \beta(a)$ for all $a \in X$. Hence $\alpha = \beta$. i.e., $\sigma_E = O$. (4) $\Longrightarrow$ (2): Clear from $\sigma_A \leq \sigma_E = O$.

DEFINITION. The automaton A is called *faithful* if one of the equivalent statements of Proposition 3 is satisfied (see Puskas [2]).

NOTE. For a set X, let $\alpha : X \to X$ be a map and let $\alpha^* : X^* \to X^*$ be the map defined by $\alpha^*(e) = e$ (empty string) and $\alpha^*(a_1 a_2 a_3 \cdots a_n) = \alpha(a_1)\alpha(a_2)\alpha(a_3)\cdots\alpha(a_n)$ for all $a_1 a_2 a_3 \cdots a_n \in X^* - \{e\}$. Then the following statements hold:

(1) α^* is bijective if α is bijective.

(2) α^* is a monoid homomorphism.

PROPOSITION 4. *Let $A = (M, X, \delta)$ be an automaton. Let $T_X = \{T_a : a \in X\}$ and let $\langle T_X \rangle$ be the semigroup generated by T_X. Then $S/\mu_M \cong \langle T_X \rangle$ where $\cong$ means semigroup-isomorphic.*

LEMMA 5. *Let $A = (M, X, \delta)$ be an automaton.*

(1) *If $f^\alpha \in \mathrm{AUT}_X(A)$, then for any $a, b \in S$,*

$$(a, b) \in \mu_M \Longleftrightarrow (\alpha^*(a), \alpha^*(b)) \in \mu_M$$

(2) *If $f^\alpha \in \mathrm{END}_X(A)$ and A is perfect, then for any $a, b \in S$,*

$$(a, b) \in \mu_M \Longrightarrow (\alpha^*(a), \alpha^*(b)) \in \mu_M.$$

Proof. For (1),

$$\begin{aligned}(a,b)\in\mu_M &\iff T_a=T_b \iff T_a(m)=T_b(m) \text{ for all } m\in M\\ &\iff ma=mb \iff f(ma)=f(mb)\\ &\iff f(m)\alpha^*(a=f(m)\alpha^*(b)\\ &\iff T_{\alpha^*(a)}(f(m))=T_{\alpha^*(b)}(f(m)) \iff T_{\alpha^*(a)}=T_{\alpha^*(b)}\\ &\iff (\alpha^*(a),\alpha^*(b))\in\mu_M.\end{aligned}$$

For (2),

$$\begin{aligned}(a,b)\in\mu_M &\iff T_a=T_b \iff T_a(m)=T_b(m) \text{ for all } m\in M\\ &\iff ma=mb \Longrightarrow f(ma)=f(mb)\\ &\iff f(m)\alpha^*(a)=f(m)\alpha^*(b)\\ &\iff T_{\alpha^*(a)}(f(m))=T_{\alpha^*(b)}(f(m)) \iff T_{\alpha^*(a)}=T_{\alpha^*(b)}\\ &\iff (\alpha^*(a),\alpha^*(b))\in\mu_M.\end{aligned}$$

LEMMA 6. *Let $A=(M,X,\delta)$ be a perfect automaton and let $\alpha,\beta: X\to X$ be maps. Let Π_α and Π_β be maps defined by $\Pi_\alpha([a])=[\alpha^*(a)]$ and $\Pi_\beta([a])=[\beta^*(a)]$ for $a\in S$ respectively where $[\,]=[\,]\mu_M$. Then for any $f^\alpha, g^\beta\in \mathrm{END}_X(A)$ the following statements hold:*
(1) Π_α and Π_β are endomorphisms.
(2) $\Pi_{\beta\alpha}=\Pi_\beta\Pi_\alpha$.
(3) $\Pi_\alpha=\Pi_\beta \iff \alpha=\beta$ if A is faithful
where the product of maps means the composition of maps.

Proof. We note that Π_α and Π_β are well-defined from lemma 5(2). For (1) and (2), it is easy to check them. For (3), for every $t\in X$, $\Pi_\alpha([t])=\Pi_\beta([t])$. This implies $[\alpha^*(t)]=[\beta^*(t)]$. Hence $(\alpha^*(t),\beta^*(t))\in\mu_M$. Since $t\in X$, $\alpha^*(t)=\alpha(t)$ and $\beta^*(t)=\beta(t)$. Moreover, $(\alpha(t),\beta(t))\in\mu_M \iff T_{\alpha(t)}=T_{\beta(t)}$. Since A is faithful, we can conclude that $T_{\alpha(t)}=T_{\beta(t)}\Longrightarrow \alpha(t)=\beta(t)$. i.e., $\alpha=\beta$. The converse is trivial.

COROLLARY 6.1. *Let $A=(M,X,\delta)$ be an automaton. Let $\alpha,\beta: X\to X$ be bijective. Then for any $f^\alpha, g^\beta\in \mathrm{AUT}_X(A)$ the following statements hold:*

(1) Π_α *and* Π_β *are semigroup-automorphisms*
(2) $\Pi_{\beta\alpha} = \Pi_\beta \Pi_\alpha$.
(3) $\Pi_\alpha = \Pi_\beta \Longleftrightarrow \alpha = \beta$ *if* A *is faithful.*

RECALL. Let S and T be semigroups. Let $f : S \to T$ be a homomorphism. The Kernel of f is the set $\operatorname{Ker} f$ of all the elements of $S \times S$ which are carried by f onto the same element of T. That is, $\operatorname{Ker} f = \{(a,b) \in S \times S : f(a) = f(b)\}$.

LEMMA 7. *Let* $A = (M, X, \delta)$ *be a perfect automaton and let* $\operatorname{End}(S/\mu_M)$ *be the set of all endomorphisms (not* X*-endomorphisms) on* S/μ_M. *Let* $h : \operatorname{END}_X(A) \to \operatorname{End}(S/\mu_M)$ *be a map defined by* $h(f^\alpha) = \Pi_\alpha$. *Then*
(1) h *is a homomorphism.*
(2) $\operatorname{Ker} h = \tau_E$ *if* A *is faithful.*

Proof. (1) is trivial. For (2), $\operatorname{Ker} h = \{(f^\alpha, g^\beta) : h(f^\alpha) = h(g^\beta)\}$. Now, from $h(f^\alpha) = h(g^\beta)$ we have $\Pi_\alpha = \Pi_\beta$. By Lemma 6(3), $\alpha = \beta$. Hence $\operatorname{Ker} h = \tau_E$.

LEMMA 8. *Let* $A = (M, X, \delta)$ *be an automaton and let* $\operatorname{Aut}(S/\mu_M)$ *be the set of all automorphisms (not* X*-automorphisms) on* S/μ_M. *Let* $h : \operatorname{AUT}_X(A) \to \operatorname{Aut}(S/\mu_M)$ *be a map defined by* $h(f^\alpha) = \Pi_\alpha$. *Then*
(1) h *is a group-homomorphism.*
(2) $\operatorname{Ker} h = \operatorname{Aut}_X(M)$ *if* A *is faithful.*

Proof. (1) is trivial. For (2), $\operatorname{Ker} h = \{f^\alpha \in \operatorname{AUT}_X(A) : h(f^\alpha) = I$ (identity map)$\}$. Now, from $\Pi_\alpha = I$ we have $\Pi_\alpha([a]) = [a]$ for all $a \in X$. This implies $[\alpha^*(a)] = [\alpha(a)] = [a]$. So, we have $[\alpha(a)] = [a] \Longleftrightarrow (\alpha(a), a) \in \mu_M \Longleftrightarrow T_{\alpha(a)} = T_\alpha \Longrightarrow \alpha(a) = a$ for all $a \in X$. Hence $\alpha = id$ and $\operatorname{Ker} h = \operatorname{Aut}_X(M)$.

From Lemma 7 and Lemma 8 we can obtain the following proposition.

PROPOSITION 9. *Let* $A = (M, X, \delta)$ *be a faithful automaton. Then*
(1) *the factor group* $\operatorname{AUT}_X(A)/\operatorname{Aut}_X(A)$ *is isomorphic to a subgroup of* $\operatorname{Aut}(S/\mu_M)$.
(2) $\operatorname{END}_X(A)/\tau_E$ *is isomorphic to a submonoid of* $\operatorname{End}(S/\mu_M)$ *if* A *is perfect.*

DEFINITION. Let $A = (M, X, \delta)$ be an automaton. Let $\Omega_M = \{f : M \to M$ is a transformation map$\}$. i.e., the semigroup of all arbitrary maps of M into M. (1) We define the *centralizer* $C(T_X)$ and the *normalizer* $N(T_X)$ of T_X in Ω_M as follows:

$$C(T_X) = \{f \in \Omega_M : T_a f = f T_a \text{ for all } T_a \in T_X\}$$
$$N(T_X) = \{f \in \Omega_M : T_X f = f T_X\}.$$

(2) We define the *permutation centralizer* (briefly $p-centralizer$) $C_p(T_X)$ and the *permutation normalizer* (briefly *p-normalizer*) $N_p(T_X)$ of T_X as follows:

$$C_p(T_X) = C(T_X) \cap S_M \text{ and } N_p(T_X) = N(T_X) \cap S_M$$

where S_M is the symmetric group over M (see Puscas [2]).

NOTE. $N(T_X)$ is a monoid and $C(T_X) \leq N(T_X)$ (a submonoid of $N(T_X)$).

LEMMA 10. *Let $A = (M, X, \delta)$ be a faithful automaton. Let $f \in N_p(T_X)$. Then for any $T_a \in T_X$ there is a unique $T_b \in T_X$ such that $fT_b = T_a f$ (or $fT_a = T_b f$).*

Proof. Suppose there is another $T_c \in T_X$ such that $T_a f = fT_c$. Then $fT_b = fT_c$ and $fT_b(m) = fT_c(m)$ for all $m \in M$. This implies that $f(mb) = f(mc)$. Since f is $1-1$, $mb = mc$. This means that $T_b(m) = T_c(m)$ for all $m \in M$. i.e., $T_b = T_c$. Hence $b = c$.

LEMMA 11. *Let $A = (M, X, \delta)$ be an automaton. Then*
(1) $\mathrm{End}_X(M) = C(T_X)$ and $\mathrm{Aut}_X(M) = C_p(T_X)$.
(2) $C_p(T_X)$ is a normal subgroup of $N_p(T_X)$.

Proof. For the first part of (1), $\mathrm{End}_X(M) \subset C(T_X)$: For any $f \in \mathrm{End}_X(M)$, it is enough to show that $fT_a = T_a f$ for all $T_a \in T_X$. To do this, choose any $m \in M$. Then $fT_a(m) = f(ma) = f(m)a = T_a f(m)$. Hence it holds. Similarly, the converse can be shown easily. The second part of (1) follows from the first part of (1). For (2), for any $f \in N_p(T_X)$, $g \in C_p(T_X)$ and $T_a \in T_X$, $T_a fgf^{-1} = fT_b gf^{-1}$ for some $T_b \in T_X = fgT_b f^{-1} = fgf^{-1}T_a$.

PROPOSITION 12. *Let $A = (M, X, \delta)$ be a faithful automaton. Then the following statements hold:*

(1) $\mathrm{AUT}_X(A) = N_p(T_X)$.

(2) $N_p(T_X)/C_p(T_X) \cong$ *a subgroup of* $\mathrm{Aut}(S/\mu_M)$.

(3) $\mathrm{Aut}_X(A)$ *is a normal subgroup of* $\mathrm{AUT}_X(A)$.

Proof. For (1), $\mathrm{AUT}_X(A) \subset N_p(T_X)$: To prove this, choose any $f \in \mathrm{AUT}_X(A)$ and let f be an αX-automorphism. Then we have $f(ma) = f(m)\alpha(a)$ for all $m \in M$ and $a \in X$. This means that $f[T_a(m)] = T_{\alpha(a)}[f(m)]$. Also, this implies that $fT_a = T_{\alpha(a)}f$. Hence since α is bijective, $fT_X = T_X f$. $\mathrm{AUT}_X(A) \supset N_p(T_X)$: By Lemma 10, for any $f \in N_p(T_X)$ and $T_a \in T_X$ $\exists!$ $T_b \in T_X$ such that $fT_a = T_b f$. Let $\alpha : X \to X$ be a map defined by $\alpha(a) = b$ with $fT_a = T_b f$. Claim: α is bijective. (i) α is well-defined: To prove this, let $t = u$ for $t, u \in X$. By Lemma 10, for T_t and $T_u \exists! T_c, T_d \in T_X$ such that $fT_t = T_c f$ and $fT_u = T_d f$. This implies $T_c f = T_d f$. Hence $T_c = T_d$. So, we have $c = d$ since X is reduced. Thus, $\alpha(t) = c = d = \alpha(u)$. (ii) $\alpha = 1-1$: Suppose $\alpha(t) = \alpha(u)$. Let $\alpha(t) = c$ with $fT_t = T_c f$ and let $\alpha(u) = d$ with $fT_u = T_d f$. Then from $c = d$ $fT_t = fT_u$. Hence $T_t = T_u$. Thus, we have $t = u$. (iii) α is onto : For any $b \in X$, consider $T_b \in T_X$. By Lemma 10 $\exists!$ $T_a \in T_X$ such that $T_b f = fT_a$. Hence $\exists$ $a \in X$ such that $\alpha(a) = b$ with $fT_a = T_b f$.

Now, we will show that f is an αX-homomorphism. For any $m \in M$ and $a \in X$,

$$\begin{aligned} f(m)\alpha(a) &= f(m)b \text{ with } fT_a = T_b f \\ &= T_b f(m) = fT_a(m) = f(ma). \end{aligned}$$

(2) follows from Proposition 9 and Lemma 11. (3) follows from Lemma 11(2).

NOTATION. Let $A = (M, X, \delta)$ be an automaton and $\alpha : S \to S$ be a map. For $m, q \in M$, $H_{m\alpha q} = \{a \in S : m\alpha(a) = q\}$ and $H_{mq} = \{a \in S : ma = q\}$.

The following lemma is a generalization of Lemma 18 of Park [1].

LEMMA 13. *Let $A = (M, X, \delta_A)$ and $B = (N, X, \delta_B)$ be automata. Let $m \in M$ be a fixed element and let $\alpha : S \to S$ be a map. If $f : M \to N$ is any map, then the following statements hold:*

(1) *If* $f(mt) = f(m)\alpha(t)$ *for all* $t \in S$, *then* $H_{mq} \subset H_{f(m)\alpha f(q)}$ *for all* $q \in M$.

(2) *If* $H_{mq} \subset H_{f(m)\alpha f(q)}$ *for some* $q \in M$, *then* $f(mt) = f(m)\alpha(t)$ *for all* $t \in H_{mq}$.

(3) $f(mt) = f(m)\alpha(t)$ *for all* $t \in H_{mq} \Longleftrightarrow H_{mq} \subset H_{f(m)\alpha f(q)}$ *for all* $q \in M$.

(4) *Assume* M *is strongly connected. Then* $f(mt) = f(m)\alpha(t)$ *for all* $t \in S \Longleftrightarrow H_{mq} \subset H_{f(m)\alpha f(q)}$ *for all* $q \in M$.

Proof. For (1), for every $a \in H_{mq}$ we have $ma = q$. This implies $f(q) = f(ma) = f(m)\alpha(a)$. Hence $a \in H_{f(m)\alpha f(q)}$. For (2), for every $t \in H_{mq}$ we have $mt = q$ and also, since $t \in H_{f(m)\alpha f(q)}$, we have $f(m)\alpha(t) = f(q)$. This implies $f(m)\alpha(t) = f(mt)$. (3) is clear from (1) and (2). For (4), suppose M is strongly connected. Then we have $M = mS$. So, for every $t \in S$, we have $k = mt$ for some $k \in M$. This implies $t \in H_{mk} \subset H_{f(m)\alpha f(k)}$. Thus, $f(m)\alpha(t) = f(k)$. Hence $f(mt) = f(k) = f(m)\alpha(t)$. The converse is clear from (1).

The following proposition is a generalization of Proposition 19 of Park [1].

PROPOSITION 14. *Let* $A = (M, X, \delta_A)$ *and* $B = (N, X, \delta_B)$ *be automata. Let* $f : M \to N$ *and* $\alpha : S \to S$ *be maps. Then the following statements are equivalent:*

(1) $f^\alpha : A \to B$ *is an* αS*-homomorphism.*
(2) $H_{mq} \subset H_{f(m)\alpha f(q)}$ *for any* $m, q \in M$.
(3) $f(qs) = f(q)\alpha(s)$ *for some* $q \in M$ *and all* $s \in S$ *if* M *is strongly connected and* α *is a semigroup-homomorphism.*

Proof. (1) $\Longrightarrow$ (2): For all $m \in M$ and $t \in S$, $f(mt) = f(m)\alpha(t)$. Hence it holds by Lemma 13(1). (2) $\Longrightarrow$ (1): To show $f(mt) = f(m)\alpha(t)$ for all $m \in M$ and $t \in S$, we recall $S = \bigcup_{q \in M} H_{mq}$ (see Proposition 11 of Park [1]). Now, for any $t \in S$, we have $t \in H_{mq}$ for some $q \in M$. By the assumption, $t \in H_{mq} \subset H_{f(m)\alpha f(q)}$. Hence it holds from (2) of Lemma 13. (2) $\Longrightarrow$ (3): Since M is strongly connected, we have $M = qS$ for some $q \in M$. This means that for any $s \in S$ there is an $k \in M$ such that $k = qs$. This implies $s \in H_{qk} \subset H_{f(q)\alpha f(k)}$ by the assumption.

Hence $f(q)\alpha(s) = f(k) = f(qs)$. (3) $\Longrightarrow$ (1): We have $M = qS$ from the strong connectedness of M. This implies that for any $m \in M$ there is an $a \in S$ such that $m = qa$. So, we have $ms = (qa)s$. Hence for any $m \in M$ and $s \in S$ we have $f(ms) = f((qa)s) = f(q(as)) = f(q)\alpha(as) = f(q)\alpha(a)\alpha(s) = [f(q)\alpha(a)]\alpha(s) = f(qa)\alpha(s) = f(m)\alpha(s)$.

COROLLARY 14.1. *Let $A = (M, X, \delta)$ be an automaton. Then $f^\alpha : M \to M$ is an αS-automorphism $\Longleftrightarrow$ f and α are permutations on M and S respectively and $H_{mq} \subset H_{f(m)\alpha f(q)}$ for any $m, q \in M$.*

The following lemma is a generalization of Lemma 1 of Park [1].

LEMMA 15. *Let $A = (M, X, \delta_A)$ and $B = (N, X, \delta_B)$ be automata. Let $\mathrm{HOM}_S(A, B)$ be the set of all αS-homomorphisms of A into B for all α's where $\alpha : S \to S$ is a map. If A is strongly connected, then for every $f^\alpha, g^\beta \in \mathrm{HOM}_S(A, B)$, $f^\alpha = g^\beta \Longleftrightarrow \alpha = \beta$ and $f(p) = g(p)$ for some $p \in M$.*

Proof. To show $f(m) = g(m)$ for all $m \in M$, from the strong connectedness of A we have $M = qS$ for all $q \in M$. This implies that $M = pS$. So, for every $m \in M$, $m = pt$ for some $t \in S$. Hence $f(m) = f(pt) = f(p)\alpha(t) = g(p)\beta(t) = g(pt) = g(m)$. The converse is trivial.

NOTE. If $f^\alpha \in \mathrm{AUT}_S(M)$, then $(f^n)^{\alpha^n} \in \mathrm{AUT}_S(M)$ for any nonnegative integer n where $f^n = fff\cdots f$ (n times) and the product means the composition of f's.

DEFINITION. Let $A = (M, X, \delta)$ be an automaton. Then we say that a mapping $\alpha : S \to S$ is an *M-homomorphism* if $m\alpha(a) = ma$ for all $m \in M$ and $a \in S$. We recall that f is a *regular permutation* on a set M if f is a permutation on M and for every power, say f^n, of f, it is the case that $f^n(p) = p$ for some $p \in M$ implies $f^n = 1$.

PROPOSITION 16. *Let $A = (M, X, \delta)$ be strongly connected and let $f^\alpha \in \mathrm{AUT}_S(M)$. Then f is a regular permutation on M if $\alpha : S \to S$ is an M-homomorphism.*

Proof. Suppose that for any $n \in N$, $f^n(x) = x$ for some $x \in M$.

Claim: $f^n = I$ (identity). (Proof). Since $f^\alpha \in \mathrm{AUT}_S(M)$, $(f^n)^{\alpha^n} \in \mathrm{AUT}_S(M)$. So, this implies $(f^n)^{\alpha^n} \in \mathrm{END}_S(M)$. Also, $I^{\alpha^n} \in \mathrm{END}_S(M)$.

We will show this. For all $m \in M$ and $a \in S$, $I(ma) = ma = m\alpha(a) = I(m)\alpha(a)$. This implies $I^\alpha \in \mathrm{AUT}_S(M)$ and $(I^n)^{\alpha^n} \in \mathrm{AUT}_S(M)$. Since $I^n = I$, we have $I^{\alpha^n} \in \mathrm{AUT}_S(M)$. Hence $I^{\alpha^n} \in \mathrm{END}_S(M)$. From Lemma 15, we can conclude $f^n = I$.

References

1. C. H. Park, *Algebraic properties associated with input semigroup S of an automaton*, Bull. Korean Math. Soc. **27** (1990), 69-83.
2. ———, *On right congruences associated with the input semigroup S of Automata*, Semigroup Forum, to appear.
3. Cs. Puskas, *On the generalized automorphism groups of automata without outputs*, Papers on automata theory vol. 1, The Department of Mathematics and Computer Science, Karl Marx University of Economics, Budapest (1980).
4. A. C. Fleck, *Isomorphism groups of automata*, J. ACM **9** (1962), 469–476.
5. R. H. Oehmke, *On right congruences of semigroups*, Semigroup Forum **16** (1978), 393-401.

Department of Mathematics
The SungHwa University
Cheon An, Korea