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CANTOR-BENDIXSON DERIVATIVES AND α-COMPACT-COVERING MAPS

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1. Introduction

A map is a continuous onto function and the notation $f : X \to Y$ denotes a map from a space X onto a space Y.

A map $f : X \to Y$ is compact-covering (resp. countable-compact-covering) if every compact (resp. countable and compact) subset of Y is the image of some compact subset of X.

Using the concepts of Cantor-Bendixson derivative (see Definition 2.1), we define (see Definition 3.1) that a map $f : X \to Y$ is *α-compact-covering* if every countable compact subset of Y whose α-th Cantor-Bendixson derivative is empty is the image of some compact subset of X.

A map $f : X \to Y$ is sequence-covering if every convergent sequence (including its limit) $S \subset Y$ is the image of some compact set (not necessarily a convergent sequence) $C \subset X$.

It follows simply from the definitions that every countable-compact-covering map is α-compact-covering, and that a sequence-covering map is 2-compact-covering in the above sense.

We now state the following theorem [**1**]:

THEOREM 1.1. *Every countable-compact-covering map $f : X \to Y$ from a separable metrizable space X onto a first-countable regular space Y with each fiber $f^{-1}(y)$ compact is compact-covering.*

This theorem solves affirmatively a question posed by Michael ([**6**], Question 1.1 (a)).

The main purpose of this paper is to prove a theorem [Theorem 3.7] that there is no countable ordinal α such that the assumption on f,

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countable-compact-covering, in Theorem 1.1 could be replaced by the assumption that f is α-compact-covering.

More precisely, for every $\alpha \in \omega_1$, there exists a map from a separable metric space X onto a countable and compact metric space Y with each fiber $f^{-1}(y)$ compact that is α-compact-covering but not $\alpha+1$-compact-covering (and hence not compact-covering, not even countable-compact-covering).

The latter theorem generalizes earlier examples by Michael [4], and Steprāns and Watson [7].

For undefined terminology, see [2].

2. Preliminaries

In this section, we will present the basic theorems needed for the rest of this paper. Throughout the remainder of the paper we assume that the spaces considered are at least Hausdorff.

DEFINITION 2.1. Let X be a topological space and α be an ordinal. The *α-th derivative* of X, denoted by $D^{(\alpha)}X$, is defined inductively as follows:

$D^{(0)}X = X$,

$D^{(\alpha+1)}X = D^{(\alpha)}X \setminus \{x : x \text{ is an isolated point in } D^{(\alpha)}X\}$,

$D^{(\alpha)}X = \bigcap_{\beta<\alpha} D^{(\beta)}X$ for limit ordinals α.

The smallest α for which $D^{(\alpha)}X = D^{(\alpha+1)}X$ is called *the Cantor-Bendixson height* of X (abbreviated CB-height in the sequel) and denoted by $CB(X)$.

EXAMPLE 2.2. If X is a finite space, then $CB(X) = 1$. If X is a space of a convergent sequence including its limit, then $CB(X) = 2$. The Cantor space C is perfect, that is, it has no isolated points. Thus $D^{(\alpha)}C = C$ for all ordinal α.

We record some useful and well-known facts about the Cantor-Bendixson height.

PROPOSITION 2.3. *Let X be a non-empty compact Hausdorff space.*

(a) *If every point of X is a cluster point of X, then X is uncountable.*

(b) *If B is a countable and compact subset of X, then $D^{(\alpha)}B = \emptyset$ for some $\alpha \in \omega_1$.*

(c) *If B is a countable and compact subset of X, then the CB-height of B is either zero or a successor ordinal $\alpha + 1$ for some $\alpha \in \omega_1$. In the latter case, $D^{(\alpha)}B$ is finite.*

(d) *If B is a countable and compact subset of X and W is open in X, then for every $\gamma \in \omega_1$, $D^{(\gamma)}(B \cap W) = D^{(\gamma)}B \cap W$.*

(e) *If B is a countable and compact subset of X and $x \in B$, then there is an open neighborhood W of x such that $D^{(\alpha)}(B \cap W) = \{x\}$ for some $\alpha \in \omega_1$.*

(f) *If E and B are countable and compact subsets of X such that $E \subset B$, then the CB-height of E is less than or equal to the CB-height of B (i.e., $CB(E) \leq CB(B)$).*

(g) *If $\mathcal{C} = \{C_i : i \in n+1\}$ is a finite collection of countable and compact subsets of X, then $D^{(\alpha)}(\bigcup_{i \in n+1} C_i) = \bigcup_{i \in n+1} D^{(\alpha)}C_i$ for every $\alpha \in \omega_1$.*

(h) *The CB-height of the union of a finite collection $\mathcal{C}$ of countable and compact subsets of X is the supremum of the CB-heights of the members of $\mathcal{C}$.*

Proof. See [**1**].

PROPOSITION 2.4. *Let Y be a first-countable Hausdorff space.*

(a) *Let $\alpha \in \omega_1$, $y' \in Y$, let $(y_n)_{n \in \omega}$ be a sequence in Y with $y_n \to y'$ such that there exists a sequence $(U_n)_{n \in \omega}$ of open neighborhoods such that each $y_n \in U_n$ and $U_n \cap U_m = \emptyset$ if $n \neq m$. Suppose $(E_n)_{n \in \omega}$ is a sequence of countable and compact sets such that $E_n \subset U_n$ and $D^{(\alpha)}E_n = \{y_n\}$ for all $n \in \omega$.*

Then $E = \bigcup_{n \in \omega} E_n \cup \{y'\}$ is countable, compact, and $D^{(\alpha+1)}E = \{y'\}$.

(b) *Let $\alpha \in \omega_1$, $y' \in Y$, and let $(O_n)_{n \in \omega}$ be a decreasing neighborhood base at y' in Y. Suppose $(E_n)_{n \in \omega}$ is a sequence of countable and compact subsets of Y such that $E_n \subset O_n$ and $D^{(\alpha)}E_n = \{y'\}$.*

Then $E = \bigcup_{n \in \omega} E_n$ is countable, compact, and $D^{(\alpha)}E = \{y'\}$.

Proof. See [**1**].

DEFINITION 2.5. Let X and Y be spaces, $A \subset X \times Y$, and $E \subset Y$. We say that E *can be lifted to a compact subset of* A if there is a compact set $B \subset A$ such that the projection π_2 of B onto the second coordinate is E. Such a B is called a *compact lifting* of E.

For $y \in Y$ and $A \subset X \times Y$, we set $A_y = \{x \in X : \langle x, y\rangle \in A\}$. We call A_y the *horizontal section* of A at y.

PROPOSITION 2.6. (a) *Let Y be a space. Suppose $K \subset Y$ and $A \subset I^\omega \times K$ are such that every countable compact subset of K can be lifted to a compact subset of A. Let $y \in E \subset K$ with E countable compact, and let O be an open neighborhood of y, $E' = E \cap O$, and let $\mathcal{G}$ be any subset of $\mathcal{H}$. Suppose for every compact lifting $D \subset A$ of E, we have $D_y \notin \mathcal{G}$.*

Then for any compact lifting $D' \subset A$ of E', we have $D'_y \notin \mathcal{G}$.

(b) *Let Y be a first-countable regular space. Let $K \subset Y$ be compact and $A \subset I^\omega \times K$ be such that every horizontal section of A is compact. Let $y \in E \subset K$ with E countable compact and $(O(y,n))_{n\in\omega}$ be a neighborhood base at y such that $E \cap O(y,n) = E \cap cl(O(y,n))$. Let $(\delta_n)_{n\in\omega}$ be a sequence of positive reals converging to zero such that for each $n \in \omega$, $E_n = E \cap (cl(O(y,n))\backslash O(y,n+1))$ can be lifted to a compact subset $D^n \subset A$ with the condition that $D^n_z \in B_{\delta_n}(A_y)$ for all $z \in E_n$.*

Then $D = cl(\bigcup_{n\in\omega} D^n)$ is a compact lifting (in A) of E.

Proof. See [**1**].

REMARK. In (b), the assumption that $E \cap O(y,n) = E \cap cl(O(y,n))$ is possible since every countable non-empty regular space is zero-dimensional (see [2], Theorem 6.2.8 and 6.2.6).

3. Main Theorem

In this section, we prove a theorem [Theorem 3.7] showing that for every $\alpha \in \omega_1$ there exists a map $f : X \to Y$ from a separable metric space X onto a metric space Y with each $f^{-1}(y)$ compact that is α-compact-covering, but not $\alpha + 1$-compact-covering.

DEFINITION 3.1. For $\alpha \in \omega_1$, a map $f : X \to Y$ from a space X onto a space Y is *α-compact-covering* if for every countable and compact $E \subset Y$ such that $D^{(\alpha)}E = \emptyset$ there is a compact $C \subset X$ such that $f[C] = E$.

PROPOSITION 3.2. *A map $f : X \to Y$ from a space X onto a Hausdorff space Y is countable-compact-covering iff f is α-compact-covering for all $\alpha \in \omega_1$.*

Proof. " Only if "-part is clear. For " if "-part, let $E \subset Y$ be countable and compact. Then $D^{(\alpha)}E = \emptyset$ for some $\alpha \in \omega_1$ by Proposition 2.3(b).

EXAMPLE 3.3. ([4], Example 4.1) There is an open map $f : X \to Y$ from a metric space onto a compact metric space Y which is not compact-covering.

REMARK. (1) The fact that the map in Example 3.3 is even countable-compact-covering was pointed out by Michael in ([5], Example 5.1).

(2) It was noted in ([3], Example 9.13) that the map in Example 3.3 is sequence-covering (every open surjection $f : X \to Y$ with first-countable X is sequence-covering. In fact, every convergent sequence in Y is the image of a convergent sequence in X).

Since the main idea in the argument of proof of Theorem 3.9 is from a proposition by Steprāns and Watson [7] and Example 3.3, we present the following results 3.4 and 3.5 as a preliminary step.

LEMMA 3.4. ([7]) *Let n be a fixed natural number, and let $A \subset [0,1]^2$ be such that for every $y \in [0,1]$ there exists an open interval $U_y \subset [0,1]$ of length $\le \frac{1}{n+1}$ such that $\{x : \langle x, y\rangle \in A\} = [0,1]\backslash U_y$. Let E be a compact subset of $[0,1]$ such that $D^{(n)}E$ is finite. Then there exists a compact $C \subset A$ such that $\pi_2[C] = E$.*

Proof. See [1].

PROPOSITION 3.5. *For every natural number n, there exist a subspace $X \subset [0,1]^2$ and a function $f : X \to [0,1]$ with each fiber $f^{-1}(y)$ compact that is n-compact-covering, but not compact-covering.*

Proof. Let π_2, B, x_y be as in Example 3.3. Let A be a space obtained from $[0,1]^2$ by removing from each horizontal interval $\pi_2^{-1}(y)$ an open interval U_y containing x_y of length $\le \frac{1}{n+1}$.

Then $A \subset B$. Let $g = \pi_2|_A$. Then $g : A \to [0,1]$ is a map with each $g^{-1}(y)$ compact (since $g^{-1}(y) = \pi_2^{-1}(x) \cap A \simeq [0,1]\backslash U_y$ is compact). Equivalently, every horizontal section ($= [0,1]\backslash U_y$) of A at y is compact.

Claim: g is n-compact-covering.

Let $E \subset [0,1]$ be a countable and compact set such that $D^{(n)}E = \emptyset$. Since E is compact, $D^{(n-1)}E$ is finite. By Lemma 3.4, there exists a compact $C \subset A$ such that $g[C] = E$. Hence g is n-compact-covering.

Claim: g is not compact-covering.

By Example 3.3, there is no compact $C \subset B$ such that $\pi_2|_B[C] = [0,1]$. Since $A \subset B$, there is no compact $D \subset A$ such that $\pi_2|_A[D] = [0,1]$. Hence $g = \pi_2|_A$ is not compact-covering.

The following Proposition is a special case of Proposition 2.4(a).

PROPOSITION 3.6. *Let $\alpha \in \omega_1$, let $(O_n)_{n\in\omega}$ be a decreasing neighborhood base at y', let $y' \in [0,1]$, let $(y_n)_{n\in\omega}$ be a one-to-one sequence in $[0,1]$ such that $y_n \to y'$, and let U_n be a neighborhood of y_n such that $U_n \subset O_n$ and $U_n \cap U_m = \emptyset$ for $n \neq m$. Suppose $(E_n)_{n\in\omega}$ is a sequence of countable and compact sets such that $E_n \subset U_n$ and $D^{(\alpha)}E_n = \{y_n\}$ for all $n \in \omega$. Then $E = \bigcup_{n\in\omega} E_n \cup \{y'\}$ is countable, compact, and $D^{(\alpha+1)}E = \{y'\}$.*

THEOREM 3.7. *For every $\alpha \in \omega_1 \backslash \{0\}$, for every $y' \in [0,1]$, and for every open neighborhood V of y', there exist a subspace $X_\alpha \subset [0,1]^2$, a countable compact subspace $E_\alpha \subset V$ with $D^{(\alpha)}E_\alpha = \{y'\}$, and a map $\pi_2|_{X_\alpha} : X_\alpha \to E_\alpha$ with each fiber $(\pi_2|_{X_\alpha})^{-1}(y)$ compact that is α-compact-covering, but E_α cannot be lifted to a compact subset of X_α (hence $\pi_2|_{X_\alpha}$ is not $\alpha + 1$-compact-covering).*

Proof. We prove this by induction on $\alpha \geq 1$.

Case 1: $\alpha = 1$. Before proceeding with the inductive argument, we fix $y' \in [0,1]$ and an open neighborhood V of y', and choose a one-to-one sequence $(y_n)_{n\in\omega}$ in V such that $y_n \to y'$.

Now let $E_1 = \{y_n : n \in \omega\} \cup \{y'\}$. Then E_1 is countable, compact, and $D^{(1)}E_1 = \{y'\}$.

Let $X_1 = \{\langle 1, y_n\rangle : n \in \omega\} \cup \{\langle 0, y'\rangle\}$.

(i) Then $\pi_2|_{X_1} : X_1 \to E_1$ is a map with each fiber $(\pi_2|_{X_1})^{-1}(y)$ compact (in fact, each fiber is a singleton set).

(ii) We claim that $\pi_2|_{X_1}$ is 1-compact-covering.

Let $E \subset E_1$ be countable and compact with $D^{(1)}E = \emptyset$. Then $D^{(0)}E = E$ is finite. If $y' \notin E$, then let $C = \{\langle 1, y\rangle : y \in E\}$. Then

C is a finite subset of X_1 and thus compact. Clearly $\pi_2|_{X_1}[C] = E$. If $y' \in E$, then let $C' = \{\langle 1, y\rangle : y \in E\backslash\{y'\}\} \cup \{\langle 0, y'\rangle\}$. Then C' is a compact subset of X_1 and $\pi_2|_{X_1}[C'] = E$.

Hence $\pi_2|_{X_1}$ is 1-compact-covering.

(iii) We claim that E_1 cannot be lifted to a compact subset of X_1 (and hence $\pi_2|_{X_1}$ is not 2-compact-covering).

By way of contradiction, assume that there exists a compact $C \subset X_1$ such that $\pi_2|_{X_1}[C] = E_1$. By the construction of X_1 and since $\pi_2|_{X_1}$ is onto, we have $C = X_1$. But X_1 is not compact since $\{\langle 1, y_n\rangle : n \in \omega\}$ is an infinite set which has no cluster point. This contradicts the assumption that C is compact.

Case 2: Suppose $\alpha = \beta + 1$ for some $\beta \in \omega_1$, and the statement is true for β.

We fix $y' \in [0, 1]$ and an open neighborhood V of y', and choose a one-to-one sequence $(y_n)_{n\in\omega}$ in V such that $y_n \to y'$ and a neighborhood base $(O_n)_{n\in\omega}$ at y'.

Let $(U_n)_{n\in\omega}$ be a sequence of neighborhoods such that each $y_n \in U_n\backslash O_n$, and $U_n \cap U_m = \emptyset$ if $n \neq m$.

By the inductive assumption, for each $n \in \omega$, there exist countable compact subsets $E_{\beta,n}$ of U_n such that $D^{(\beta)}E_{\beta,n} = \{y_n\}$, and subsets $X_{\beta,n}$ of $[0,1]^2$ such that $\pi_2|_{X_{\beta,n}} : X_{\beta,n} \to E_{\beta,n}$ is a map with each fiber $(\pi_2|_{X_{\beta,n}})^{-1}(y)$ compact that is β-compact-covering, but $E_{\beta,n}$ cannot be lifted to a compact subset of $X_{\beta,n}$ (and hence $\pi_2|_{X_{\beta,n}}$ is not $\beta + 1$-compact-covering, i.e., not α-compact-covering).

We can multiply the first coordinates of points in each $X_{\beta,n}$ by $\frac{1}{2}$ to obtain a homeomorphic image of $X_{\beta,n}$ in $[0, \frac{1}{2}] \times U_n$ which has the properties stated for $\pi_2|_{X_{\beta,n}}$. For simplicity of notation, we will denote this homeomorphic image by $X_{\beta,n}$ again.

Let $E_\alpha = (\bigcup_{n\in\omega} E_{\beta,n}) \cup \{y'\}$.

Then, by Proposition 3.6, E_α is countable, compact, and $D^{(\alpha)}E_\alpha = \{y'\}$.

Now let

$$X_\alpha = (\bigcup_{n\in\omega} X_{\beta,n}) \cup (\bigcup_{n\in\omega} \{1\} \times E_{\beta,n}) \cup ([0, \frac{1}{2}] \times \{y'\}).$$

Then X_α is a subspace of $[0,1]^2$. Let $f = \pi_2|_{X_\alpha}$.

Then $f : X_\alpha \to E_\alpha$ is a map and f is an extension of each $\pi_2|_{X_{\beta,n}}$.

(i) We claim that each fiber $f^{-1}(y)$ is compact.

If $y \in E_\alpha \backslash \{y'\}$, then there is some $n \in \omega$ such that $y \in E_{\beta,n}$. Hence $f^{-1}(y) = (\pi_2|_{X_{\beta,n}})^{-1}(y) \cup \{\langle 1, y\rangle\}$. Since $(\pi_2|_{X_{\beta,n}})^{-1}(y)$ is compact by the inductive assumption, $f^{-1}(y)$ is compact.

If $y = y'$, then $f^{-1}(y) = [0, \frac{1}{2}]$ is clearly compact.

(ii) We claim that f is α-compact-covering.

Let $E \subset E_\alpha$ be countable and compact with $D^{(\alpha)}E = \emptyset$.

Subcase i) If y' is not a cluster point of E, then there exists an open neighborhood U of y' such that $U \cap (E \backslash \{y'\}) = \emptyset$. So $E\backslash\{y'\}$ is covered by finitely many $E_{\beta,n}$'s and so there exists m such that $E\backslash\{y'\} \subset \bigcup_{0 \leq n \leq m} E_{\beta,n}$. Let $C = \{1\} \times \bigcup_{0 \leq n \leq m}(E_{\beta,n} \cap E)$. Then C is clearly a compact subset of X_α.

If $y' \notin E$, then $f[C] = E$. If $y' \in E$, then $C \cup ([0, \frac{1}{2}] \times \{y'\})$ is a compact subset of X_α such that $f[C \cup ([0, \frac{1}{2}] \times \{y'\})] = E$

Subcase ii) Suppose y' is a cluster point of E. Since $D^{(\alpha)}E = \emptyset$, we have $D^{(\beta)}E$ is finite. Note that for all $n \in \omega$, $D^{(\beta)}(E \cap E_{\beta,n}) \subset D^{(\beta)}E$ and so $\bigcup_{n\in\omega} D^{(\beta)}(E \cap E_{\beta,n}) \subset D^{(\beta)}E$.

Since the sets $E \cap E_{\beta,n}$ are pairwise disjoint and since $D^{(\beta)}E$ is finite, $D^{(\beta)}(E \cap E_{\beta,n}) = \emptyset$ for all but finitely many n's. But since for all $n \in \omega$, $D^{(\beta)}(E \cap E_{\beta,n}) \subset D^{(\beta)}E_{\beta,n} = \{y_n\}$, we may assume that there is some $i \in \omega$ such that if $m \geq i$, then $D^{(\beta)}(E \cap E_{\beta,m}) = \emptyset$; but if $n < i$, then $D^{(\beta)}(E \cap E_{\beta,n}) = \{y_n\}$.

By the inductive assumption (β-compact-covering) applied to the case $D^{(\beta)}(E \cap E_{\beta,m}) = \emptyset$ above, there exists a compact set $C_m \subset X_{\beta,m}$ such that $\pi_2|_{X_{\beta,m}}[C_m] = E \cap E_{\beta,m}$. Then C_m is also compact in X_α and $f[C_m] = E \cap E_{\beta,m}$.

Let $C' = cl(\bigcup_{m \geq i} C_m)$. Then C' is clearly compact. Also by the same argument as used in the proof of Proposition 2.6 (b) (with the fact that y' is a cluster point and $[0, \frac{1}{2}]$ is compact), we get $C' \subset \bigcup_{m\geq i} C_m \cup ([0, 1] \times \{y'\})$, and $f[C'] = \bigcup_{m\geq i}(E \cap E_{\beta,m}) \cup \{y'\}$. Also for every $n < i$, y' is not a cluster point of $E \cap E_{\beta,n}$ and so as in the previous case, there exists a compact set $C_n \subset X_\alpha$ such that $f[C_n] = E \cap E_{\beta,n}$. Let $C'' = \bigcup_{n<i} C_n$. Then C'' is a compact subset of X_α such that $f[C''] = \bigcup_{n<i}(E \cap E_{\beta,n})$ Let $C = C' \cup C''$. Then C is a compact subset of X_α and $f[C] = E$.

Hence f is α-compact-covering.

(iii) We claim that E_α cannot be lifted to a compact subset of X_α (and hence f is not $\alpha+1$-compact-covering).

Recall from the construction that E_α is countable and compact with $D^{(\alpha)}E_\alpha = \{y'\}$. Then $D^{(\alpha+1)}E_\alpha = \emptyset$ and for every $n \in \omega$, $D^{(\alpha)}E_{\beta,n} = \emptyset$. By the inductive assumption, each $E_{\beta,n}$ cannot be lifted by $\pi_2|_{X_{\beta,n}} = f|_{X_{\beta,n}}$ to a compact subset of $X_{\beta,n}$.

By way of contradiction, suppose that there exists a compact set $C \subset X_\alpha$ such that $f[C] = E_\alpha$. Since each $E_{\beta,n}$ cannot be lifted to by f to $X_{\beta,n}$, and since f is onto, there exists $y'_n \in E_{\beta,n}$ with $\langle 1, y'_n\rangle \in (\{1\} \times E_{\beta,n}) \cap C$ for all $n \in \omega$. So we have a countably infinite set $\{\langle 1, y'_n\rangle : n \in \omega\}$ in C which has no cluster point (the only possible candidate for a cluster point is $\{\langle 1, y'\rangle\}$, but it is not in X_α by the construction). Hence C is not countably compact and therefore not compact. This contradicts the assumption that C is compact.

Case 3: Suppose α is a limit ordinal and assume that the statement is true for all $\beta < \alpha$.

Let $y' \in [0,1]$ and V be an open neighborhood of y', and let $(y_n)_{n\in\omega}$ be a one-to-one sequence in V such that $y_n \to y'$ and a neighborhood base $(O_n)_{n\in\omega}$ at y'. Let $(U_n)_{n\in\omega}$ be a sequence of open neighborhoods such that each $y_n \in U_n \subset O_n$, and $U_n \cap U_m = \emptyset$ if $n \neq m$.

Since α is a limit ordinal, there is a strictly increasing sequence $(\alpha_n)_{n\in\omega}$ of ordinals in ω_1 such that for all $n \in \omega$, $\alpha_n < \alpha$ and for all $\gamma < \alpha$, there exists $n' \in \omega$ with $\gamma < \alpha_{n'} < \alpha$.

By the inductive assumption, for each $n \in \omega$, there exist countable compact sets $E_{\alpha_n} \subset U_n$ such that $D^{(\alpha_n)}E_{\alpha_n} = \{y_n\}$, and subsets $X_{\alpha_n} \subset [0,1]^2$ such that $\pi_2|_{X_{\alpha_n}} : X_{\alpha_n} \to E_{\alpha_n}$ is a map with each fiber $(\pi_2|_{X_{\alpha_n}})^{-1}(y)$ compact that is α_n-compact-covering, but E_{α_n} cannot be lifted to a compact subset of X_{α_n} (and hence $\pi_2|_{X_{\alpha_n}}$ is not α_n+1-compact-covering).

By multiplying the first coordinates of points in each X_{α_n} by $\frac{1}{2}$, we obtain a homeomorphic image of X_{α_n} in $[0,\frac{1}{2}] \times U_n$ which has the properties stated for $\pi_2|_{X_{\alpha_n}}$.

For simplicity of notation, we will denote this homeomorphic image by X_{α_n} again.

Let $E_\alpha = \bigcup_{n\in\omega} E_{\alpha_n} \cup \{y'\}$. Then, by Proposition 3.6, E_α is countable and compact.

We claim that $D^{(\alpha)}E_\alpha = \{y'\}$.

It suffices to show that:

(i) if $y \in E_\alpha \backslash \{y'\}$, then $y \notin D^{(\alpha)}E_\alpha$;

(ii) $y' \in \bigcap_{n\in\omega} D^{(\alpha_n+1)}E_\alpha$; and

(iii) $\bigcap_{n\in\omega} D^{(\alpha_n+1)}E_\alpha = D^{(\alpha)}E_\alpha$.

For (i), let $y \in E_\alpha \backslash \{y'\}$. Then there exists $n \in \omega$ such that $y \in E_{\alpha_n} \subset U_n$. By way of contradiction, we assume that $y \in D^{(\alpha)}E_\alpha$. Then $y \in D^{(\alpha_n+1)}E_\alpha$. This means that every deleted open neighborhood of y contains points of $D^{(\alpha_n+1)}E_\alpha$. Since $D^{(\alpha_n)}E_{\alpha_n} = \{y_n\}$ for each $n \in \omega$, we have $D^{(\alpha_n+1)}E_{\alpha_n} = \emptyset$. So U_n contains no points of $D^{(\alpha_n+1)}E_{\alpha_n}$. Also, since $U_n \cap E_{\alpha_m} = \emptyset$ for $n \neq m$, U_n contains no points of $D^{(\alpha_n+1)}E_{\alpha_m}$. But since $y \neq y'$, U_n contains no points of $D^{(\alpha_n+1)}E_\alpha$, which is a contradiction.

For (ii), let $n \in \omega$ be fixed. Suppose W is an open neighborhood of y'. Since $y_n \to y'$, there exists $n_0 > n$ such that $m \geq n_0$ implies that $y_m \in W$. Since $y_m \in D^{(\alpha_m)}E_{\alpha_m} \subset D^{(\alpha_m)}E_\alpha \subset D^{(\alpha_{n_0})}E_\alpha \subset D^{(\alpha_n)}E_\alpha$ for all $m \geq n_0$, it follows that $y' \in D^{(\alpha_n+1)}E_\alpha$. Hence $y' \in \bigcap_{n\in\omega} D^{(\alpha_n+1)}E_\alpha$.

For (iii), $D^{(\alpha)}E_\alpha = \bigcap_{\beta<\alpha} D^{(\beta)}E_\alpha \subset \bigcap_{n\in\omega} D^{(\alpha_n+1)}E_\alpha$, and since for all $\beta < \alpha$, there exists $n \in \omega$ such that $\beta < \alpha_n + 1$ and $D^{(\alpha_n+1)}E_\alpha \subset D^{(\beta)}E_\alpha$, it follows that $\bigcap_{n\in\omega} D^{(\alpha_n+1)}E_\alpha \subset D^{(\alpha)}E_\alpha$. Hence $D^{(\alpha)}E_\alpha = \bigcap_{n\in\omega} D^{(\alpha_n+1)}E_\alpha$.

Now let

$$X_\alpha = (\bigcup_{n\in\omega} X_{\alpha_n}) \cup (\bigcup_{n\in\omega} \{1\} \times E_{\alpha_n}) \cup ([0, \frac{1}{2}] \times \{y'\}).$$

Then X_α is a subspace of $[0,1]^2$. Let $f = \pi_2|_{X_\alpha}$.

Then $f : X_\alpha \to E_\alpha$ is a map and f is an extension of each $\pi_2|_{\tilde{X}_{\alpha_n}}$.

(i) We claim that each fiber $f^{-1}(y)$ is compact.

The proof is the same as the proof of Case 2(i).

(ii) We claim that f is α-compact-covering.

Let $E \subset E_\alpha$ be countable and compact with $D^{(\alpha)}E = \emptyset$.

If y' is not a cluster point of E, then by the same argument as used in Case 2, E can be lifted to a compact subset of E_α.

Suppose y' is a cluster point of E. Since $D^{(\alpha)}E = \bigcap_{\beta<\alpha} D^{(\beta)}E = \emptyset$, there exists $\alpha_n < \alpha$ with $D^{(\alpha_n+1)}E = \emptyset$ and so $D^{(\alpha_n)}E$ is finite. As in the proof of subcase ii) of Case 2, we may assume that there is some $i \in \omega$ such that if $m \geq i$, then $D^{(\alpha_m)}(E \cap E_{\alpha_m}) = \emptyset$; but if $n < i$, then $D^{(\alpha_n)}(E \cap E_{\alpha_n}) = \{y_n\}$.

By the inductive assumption (α_m-compact-covering) applied to the case $D^{(\alpha_m)}(E \cap E_{\alpha_m}) = \emptyset$ above, for each $m \geq i$ there exists a compact set $C_m \subset X_{\alpha_m}$ such that $\pi_2|_{X_{\alpha_m}}[C_m] = E \cap E_{\alpha_m}$. Then C_m is also compact in X_α and $f[C_m] = E \cap E_{\alpha_m}$.

Let $C' = cl(\bigcup_{m\geq i} C_m)$. Then C' is clearly compact. Also by the same argument as in the proof of Case 2, $C' \subset \bigcup_{m\geq i} C_m \cup ([0,1] \times \{y'\})$, and $f[C'] = \bigcup_{m\geq i}(E \cap E_{\alpha_m}) \cup \{y'\}$. Also for every $n < i$, y' is not a cluster point of $E \cap E_{\alpha_n}$ and so as in the previous case, there exists a compact set $C_n \subset X_\alpha$ such that $f[C_n] = E \cap E_{\alpha_n}$. Let $C'' = \bigcup_{n<i} C_n$. Then C'' is a compact subset of X_α such that $f[C''] = \bigcup_{n<i}(E \cap E_{\alpha_n})$ Let $C = C' \cup C''$. Then C is a compact subset of X_α and $f[C] = E$. Hence f is α-compact-covering.

(iii) As in the proof of Case 2(iii), we can show that E_α cannot be lifted to a compact subset of X_α (and hence f is not $\alpha + 1$-compact-covering).

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