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# ON FRÉCHET SPACES AND SEQUENTIAL CONVERGENCE GROUPS

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## 1. Introduction

In [5], J. Novăk introduced the notions of Cauchy sequences and completeness for convergence commutative groups and obtained a completion of these groups. Recently, the author, in [4], introduced sequential convergence spaces which have relations with other spaces specified by the knowledge of their convergence sequence as follows:

$$\text{Fréchet spaces[1]} \Longrightarrow \text{sequential convergence spaces},$$

$$\text{Hausdorff sequential convergence spaces} \Longrightarrow \text{convergence spaces [6]}.$$

Section 2 is concerned with sequential convergence spaces and their properties and their relations to Fréchet spaces. Section 3 is devoted to an investigation of products of sequential convergence spaces. In the final section, we introduce sequential convergence groups and we obtain a completion of these groups satisfying given condition $(**)$.

## 2. Sequential convergence spaces and Fréchet spaces

In this section, we shall introduce sequential convergence structures and sequential convergence spaces, and we shall show relationships between sequential convergence spaces and Fréchet spaces.

*Definition 2.1 [1].* A topological space $X$ is *Fréchet*(also called *Fréchet-Urysohn* [7]) if every point in the closure of a subset $A$ of $X$ is a limit of a sequence of points of $A$.

*Definition 2.2 [4].* Let $X$ be a non-empty set and let $S[X]$ be the set of all sequences in $X$. A non-empty subfamily $L$ of $S[X] \times X$ is

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called *a sequential convergence structure on* $X$ if it satisfies the following properties:

(SC 1) For each $x \in X$, $((x), x) \in L$, where $(x)$ is the constant sequence whose $n$-th term is $x$ for all indices $n \in N$.

(SC 2) If $((x_n), x) \in L$, then $((x_{i_n}), x) \in L$ for each subsequence $(x_{i_n})$ of $(x_n)$.

(SC 3) Let $x \in X$ and $A \subset X$. If $((x_n), x) \notin L$ for each $(x_n) \in S[A]$, then $((y_n), x) \notin L$ for each

$$(y_n) \in S[\{y \in X \mid ((x_n), y) \in L \text{ for some } (x_n) \in S[A]\}].$$

If a sequential convergence structure $L$ on $X$ is given, the pair $(X, L)$ is called *a sequential convergence space*. Hereafter, we use the notation $SC[X]$ for the set of all sequential convergence structures on $X$ and $x_1, x_2, x_3, \ldots \longrightarrow x$ will be used also for $((x_n), x) \in L$.

Let $(X, \mathcal{T})$ be a Fréchet space and let $L_{\mathcal{T}}$ denote the set of all pairs $((x_n), x) \in S[X] \times X$ such that $(x_n)$ converges to $x$ in the space $(X, \mathcal{T})$. Then, it is clear that $L_{\mathcal{T}} \in SC[X]$, and two topological spaces $(X, \mathcal{T})$ and $(X, L_{\mathcal{T}})$ are precisely same since $(X, \mathcal{T})$ is a Fréchet space. Hence we have that every Fréchet space is a sequential convergence space. And, for each $L \in SC[X]$, define a mapping $c_L$ of the power set $\mathcal{P}(X)$ of $X$ into itself as follows:

$$c_L(A) = \{x \in X \mid ((x_n), x) \in L \text{ for some } (x_n) \in S[A]\}.$$

Then $c_L$ is a Kuratowski closure operator on $X$ and $(X, c_L)$ is a Fréchet space. Let $\mathcal{L}(c_L)$ denote the set of all pairs $((x_n), x) \in S[X] \times X$ such that $(x_n)$ converges to $x$ in the space $(X, c_L)$. By the following example, we see that $L \subsetneqq \mathcal{L}(c_L)$, in general. Hence we have that every sequential convergence space $(X, L)$ need not be a Fréchet(topological) space, even if $(X, L)$ determines a Fréchet space $(X, c_L)$ as above.

*Example 2.3 [4].* Let $Q$ be the set of all rational numbers and let $L = \{((x), x) | x \in Q\} \cup \{((x_n), x) \in S[Q] \times Q \mid (x_n)$ converges to $x$ in $Q$ with the usual topology and $(x_n)$ is either increasing or decreasing$\}$. Then, $L \in SC[Q]$, but $L \subsetneqq \mathcal{L}(c_L) = \{((x_n), x) \in S[Q] \times Q \mid (x_n)$ converges to $x$ in $Q$ with the usual topology$\}$.

THEOREM 2.4 [4]. *There exists an one-to-one correspondence between the set of all Fréchet topologies on a set $X$ and $\{c_L \mid L \in SC[X]\}$.*

## 3. Products of sequential convergence spaces

In this section, we shall show that the product of two sequential convergence spaces need not be a sequential convergence space, and we shall give a sufficient condition for the product of two sequential convergence spaces to be a sequential convergence space.

*Example 3.1.* Let $L = R/Z$, the real line(equipped with the usual topology) with the integers identified, and let $I = [0,1]$ the closed unit interval in the real line. In [3], S. P. Franklin shown that $X$ and $I$ are Fréchet spaces,but $X \times I$ is not. We now show that $X \times I$ is not a sequential convergence space. For each $n, k \in N$, let $z_{nk} = (n - \frac{1}{k+1}, \frac{1}{n})$, and let $A = \{z_{nk} \mid n, k \in N\}$. Then it is easy to verify that for each $n \in N$, the sequence $(z_{nk})$ converges to $(0, \frac{1}{n})$ in $X \times I$, and the sequence $((0, \frac{1}{n}))$ converges to $(0,0)$ in $X \times I$. But, there does not exit a sequence $(z_{nk})$ in $A$ such that $(z_{nk})$ converges to $(0,0)$. Therefore, $X \times I$ is not a sequential convergence space.

Now, we shall show that the following condition $(*)$ is sufficient for the product of two sequential convergence spaces to be a sequential convergence space.

$(*)$ Let $((x_n), x) \in L$ and let $((x_{nm}), x_n) \in L$ for each $n \in N$. It is possible to choose a cross-sequence $(x_{nm(n)})$ in the double sequence $(x_{nm})$ such that (1) $((x_{nm(n)}), x) \in L$, (2) $m(n) \geqq n$ for all $n \in N$ and (3) $((x_{nk(n)}), x) \in L$ if $k(n) \geqq m(n)$ for all $n \in N$.

It is clear that the condition $(*)$ implies (SC 3) and $(*)$ is more stronger than the condition $(*)$ in [7]. The following examples show that every sequential convergence space and every Fréchet space need not satisfy $(*)$, in general, and there are Fréchet spaces and sequential convergence spaces satisfying $(*)$.

*Example 3.2.* (1) The real line $R$ is a Fréchet space(also a sequential convergence space) satisfying $(*)$.

(2) The space $(Q, L)$ as in Example 2.3 does not satisfy $(*)$.

(3) Let $Q/Z$, the set of all rational numbers(equipped with the usual topology) with the integers identified. Then $Q/Z$ is a Fréchet space by 2.3 Proposition in [2]. But, it does not satisfy $(*)$. Let $x_{nm} = n - \frac{1}{m+1}$ for each $n, m \in N$. Then, for each $n \in N$, $(x_{nm})$ converges to 0 in $Q/Z$. Since every cross-sequence $(x_{nm(n)})$ with $m(n) \geqq n$ for all $n \in N$ is divergent, hence we have that $Q/Z$ does not satisfy $(*)$.

(4) Let $L_Q = \{((x_n, x) \in S[Q] \times R | (x_n)$ converges to $x$ in the real line $R\}$. Then $(R, L_Q)$ is a sequential convergence space satisfying $(*)$, but not a Fréchet space.

THEOREM 3.3. *Let $(X, L_X)$ and $(Y, L_Y)$ be any two sequential convergence spaces satisfying $(*)$ and let*

$$L_X \times L_Y = \{((x_n, y_n), (x, y)) | ((x_n), x) \in L_X \text{ and } ((y_n), y) \in L_Y\}.$$

*Then $(X \times Y, L_X \times L_Y)$ is a sequential convergence space satisfying $(*)$.*

*Proof.* It is clear that $L_X \times L_Y$ satisfies (SC 1) and (SC 2). Since $(*)$ implies (SC 3), it is enough to show that $L_X \times L_Y$ satisfies $(*)$. Let $((x_n, y_n), (x, y)) \in L_X \times L_Y$ and let $((x_{nm}, y_{nm}), (x_n, y_n)) \in L_X \times L_Y$ for each $n \in N$. Then, by definition of $L_X \times L_Y$, we have that $((x_n), x) \in L_X$, $((y_n), y) \in L_Y$, $((x_{nm}, x_n) \in L_X$ for each $n \in N$ and $((y_{nm}), y_n) \in L_Y$ for each $n \in N$. Since $(X, L_X)$ and $(Y, L_Y)$ satisfy $(*)$, there are two cross-sequences $(x_{nm(n)})$ and $(y_{nl(n)})$ in the double sequences $(x_{nm})$ and $(y_{nm})$, respectively, such that (1) $((x_{nm(n)}), x) \in L_X$ and $((y_{nl(n)}), y) \in L_Y$, and these cross-sequences satisfy also the properties (2) and (3) of $(*)$, respectively. Let $p(n) = max\{m(n), l(n)\}$, for each $n \in N$. Then $((x_{np(n)}), x) \in L_X$ and $((y_{np(n)}), y) \in L_Y$, and hence we obtain a cross-sequence $(x_{np(n)}, y_{np(n)})$ in the double sequence $(x_{nm}, y_{nm})$ such that (1) $((x_{np(n)}), y_{np(n)}), (x, y)) \in L_X \times L_Y$, and this cross-sequence satisfies also the remain properties (2) and (3) of $(*)$.

Note that If $(X, L)$ is a sequential convergence space satisfying $(*)$, by above theorem, we see that $(X \times X, \mathcal{L}(c_L) \times \mathcal{L}(c_L))$ is a sequential convergence space satisfying $(*)$, but the product need not be a Fréchet space even if $(X, L)$ is a Fréchet space satisfying $(*)$. Indeed, $\mathcal{L}(c_{L \times L}) \neq \mathcal{L}(c_L) \times \mathcal{L}(c_L)$,in general, even if $(X, L)$ is a Fréchet space satisfying $(*)$.

## 4. Sequential convergence groups

In this section, the notions of sequential convergence groups and completeness for sequential convergence groups are introduced, and we shall obtain a completion of sequential convergence groups.

A sequential convergence space $(X, L)$ is called *Hausdorff* if $L$ satisfies the following property:

(SC 4) If $((x_n), x) \in L$ and $((x_n), y) \in L$, then $x = y$.

*Definition 4.1.* Let $(X, L)$ be a Hausdorff sequential convergence space satisfying $(*)$ and let $\cdot$ be a commutative group operator on $X$. The triple $(X, \cdot, L)$ is called *a sequential convergence group* if it satisfies the following property:

(SCG) For each $((x_n), x), ((y_n), y) \in L, ((x_n y_n^{-1}), xy^{-1}) \in L$.

*Remark.* Define a mapping $\phi$ of $X \times X$ onto $(X, \cdot, L)$ by taking $\phi(x, y) = xy^{-1}$, for each $(x, y) \in X \times X$. Then, it is obvious that (SCG) is equivalent to the following

(SCG') For each $((x_n, y_n), (x, y)) \in L \times L, ((\phi(x_n, y_n), \phi(x, y)) \in L$; that is, $\phi$ is (sequential)continuous.

Let $(X, \cdot, L)$ be a sequential convergence group. A sequence $(x_n) \in S[X]$ is called *Cauchy* if for each subsequences $(x_{i_n})$ and $(x_{j_n})$ of $(x_n)$, $((x_{i_n} x_{j_n}^{-1}), \mathbf{e} \in L$, where $\mathbf{e}$ is the identity element of the group $(X, \cdot)$. This is inspired by J. Novǎk [5]. Let $C[X]$ denote the set of all Cauchy sequences in $(X, \cdot, L)$. A sequential convergence group $(X, \cdot, L)$ is called *complete* if for each $(x_n) \in C[X], ((x_n), x) \in L$ for some $x \in X$.

Let $(X, \cdot, L)$ be a sequential convergence group and let $\sim$ be an equivalence relation on $C[X]$ defined by $(x_n) \sim (y_n)$ if and only if $((x_{i_n} y_{j_n}^{-1}), e) \in L$ for each subsequence $(x_{i_n})$ of $(x_n)$ and each subsequence $(y_{j_n})$ of $(y_n)$. The class of all Cauchy sequences which are equivalent to an $(x_n) \in C[X]$ will be denoted by $[(x_n)]$; in particular, for each constant sequence $(x)$, $x$ will be used for the equivalence class $[(x)]$. Let $X^* = \{[(x_n)] | (x_n) \in C[X]\}$ and let $\phi : X \to X^*$ denote the mapping defined by $\phi(x) = x$ for all $x \in X$. Then, it is clear that $\phi$ is injective, and if $(X, \cdot, L)$ is complete, then $\phi$ is bijective.

LEMMA 4.2. *Let $(X, \cdot, L)$ be a sequential convergence group. Then we have the following*

(1) $((x_n), x) \in L$ *if and only if* $(x_n) \in C[X]$ *and* $(x_n) \in x$.

(2) *If* $(x_n) \in C[X]$ *and* $(y_n) \in C[X]$, *then* $(x_n y_n) \in C[X]$.

(3) *If* $(x_n) \in C[X]$ *and* $(y_n) \in S[X]$ *with* $((x_n y_n^{-1}), \mathbf{e}) \in L$, *then* $(y_n) \in C[X]$ *and* $(y_n) \in [(x_n)]$.

*Proof.* (1) Let $((x_n), x) \in L$. Then, by (SC 2), $((x_{i_n}), x) \in L$ for each subsequence $(x_{i_n})$ of $(x_n)$. Hence $((x_{i_n} x_{j_n}^{-1}), \mathbf{e}) \in L$ for each subsequences $(x_{i_n})$ and $(x_{j_n})$ of $(x_n)$, and so $(x_n) \in C[X]$. Since $((x_n), x) \in L$, it is clear that $(x_n) \in x$.

Conversely, let $(x_n) \in C[X]$ with $(x_n) \in x$. Then, for each subsequence $(x_{i_n})$ of $(x_n)$, $((x x_{i_n}^{-1}), x) \in L$ since $(x_n) \in x$, and hence we have $((x x_n^{-1}), \mathbf{e}) \in L$. It follows that $((x_n), x) \in L$ since $((x), x) \in L$ by (SC 1).

(2) Let $(x_n) \in C[X]$ and $(y_n) \in C[X]$. Then for each subsequences $(x_{i_n})$ and $(x_{j_n})$ of $(x_n)$ and each subsequences $(y_{i_n})$ and $(y_{j_n})$ of $(y_n)$, $((x_{i_n} x_{j_n}^{-1}), \mathbf{e}) \in L$ and $((y_{i_n} y_{j_n}^{-1}), \mathbf{e}) \in L$. Hence we have that

$$(((x_{i_n} x_{j_n}^{-1})(y_{i_n} y_{j_n}^{-1})), \mathbf{e}) = (((x_{i_n} y_{i_n})(x_{j_n} y_{j_n})^{-1}), \mathbf{e}) \in L,$$

and so $(x_n y_n) \in C[X]$.

(3) Let $(x_n) \in C[X]$ and $(y_n) \in S[X]$ with $((x_n y_n^{-1}), \mathbf{e}) \in L$. Then, for each subsequences $(x_{i_n} y_{i_n}^{-1})$ and $(x_{j_n} y_{j_n}^{-1})$ of $(x_n y_n^{-1})$, we have that $((x_{i_n} y_{i_n}^{-1}), \mathbf{e}) \in L$ and $((x_{j_n} y_{j_n}^{-1}), \mathbf{e}) \in L$ by (SC 2). Since

$$\begin{aligned} y_{i_n} y_{j_n}^{-1} &= (x_{i_n}^{-1} y_{i_n})(x_{j_n} y_{j_n}^{-1})(x_{i_n} x_{j_n}^{-1}), \\ x_{i_n} y_{j_n}^{-1} &= (x_{i_n} x_{j_n}^{-1})(x_{j_n} y_{j_n}^{-1}) \end{aligned}$$

and

$$(x_n) \in C[X],$$

we have that

$$((x_{i_n} y_{j_n}^{-1}), \mathbf{e}) \in L \text{ and } ((y_{i_n} y_{j_n}^{-1}), \mathbf{e}) \in L.$$

Hence $(y_n) \in C[X]$ and $(y_n) \in [(x_n)]$.

By Lemma 4.2(2), we can define an (group) operator $*$ on $X^*$ as follows: for each $[(x_n)], [(y_n)] \in X^*$, $[(x_n)] * [(y_n)] = [(x_n y_n)]$. Then we obtain immediately the following theorem, and hence we omit the proof.

THEOREM 4.3. *Let $(X, \cdot, L)$ be a sequential convergence group. Then, $(X^*, *)$ is a commutative group containing $(X, \cdot)$ as a subgroup (in the sense of the group-isomorphic).*

Now we will construct a sequential convergence structure $L^*$ on $X^*$. Let $L^*$ be the set of all pairs $((\alpha_n), \alpha) \in S[X^*] \times X^*$ satisfying the condition that there exists $(x_n) \in C[X]$ such that $\alpha_m x_m^{-1} = \alpha[(x_n)]^{-1}$ for each $m \in N$. Then we have the following:

LEMMA 4.4. (1) *For each $((x_n), x) \in L$, $((x_n), x) \in L^*$.*
(2) *For each $(x_n) \in C[X]$, $((x_n), [(x_n)]) \in L^*$.*
(3) *$L^*$ satisfies (SC 1) and (SC 2).*
(4) *If $((\alpha_n), \alpha) \in L^*$, then $\alpha_n \alpha_m^{-1} \in X$ for each $n, m \in N$.*
(5) *If $((\alpha_n), \alpha) \in L^*$ and $((\beta_n), \beta) \in L^*$, then $((\alpha_n \beta_n^{-1}), \alpha\beta^{-1}) \in L^*$, i.e., $L^*$ satisfied (SCG).*

*Proof.* (1), (2) and (3) are clear.
(4) Since $((\alpha_n), \alpha) \in L^*$, there exists $(x_n) \in C[X]$ such that

$$\alpha_m x_m^{-1} = \alpha[(x_n)]^{-1} \text{ for each } m \in N.$$

Hence we have that

$$(\alpha_n x_n^{-1})(\alpha_m x_m^{-1})^{-1} = \mathbf{e} \text{ for each } n, m \in N,$$

and thus $\alpha_n \alpha_m^{-1} = x_n x_m^{-1} \in X$ for each $n, m \in N$.
(5) Since $((\alpha_n), \alpha) \in L^*$ and $((\beta_n), \beta) \in L^*$, there are $(x_n) \in C[X]$ and $(y_n) \in C[X]$ such that $\alpha_m x_m^{-1} = \alpha[(x_n)]^{-1}$ for each $m \in N$ and $\beta_m y_m^{-1} = \beta[(y_n)]^{-1}$ for each $m \in N$. Hence, $(\alpha_m \beta_m^{-1})(x_m y_m^{-1})^{-1} = (\alpha\beta^{-1})[(x_n y_n^{-1})]^{-1}$ for each $m \in N$, and thus $((\alpha_n \beta_n^{-1}), \alpha\beta^{-1}) \in L^*$ since $(x_n y_n^{-1}) \in C[X]$ by Lemma 4.2(2).

LEMMA 4.5. *Let $((\alpha_n), \alpha) \in L^*$ and let $(x_{nm})$ be a double sequence in $X$ with $[(x_{nm})] = \alpha_n$ for each $n \in N$. Then, there exists a cross-sequence $(x_{nm(n)})$ in $(x_{nm})$ such that (1)$[(x_{nm(n)})] = \alpha$, (2)$m(n) \geqq n$ for all $n \in N$ and (3)$[(x_{nk(n)})] = \alpha$ if $k(n) \geqq m(n)$ for all $n \in N$.*

*Proof.* First, we prove this lemma in particular case: $\alpha_n = \alpha$ for each $n \in N$. By Lemma 4.4(5), we obtain a double sequence $(x_{nm} x_{1m}^{-1})$ with

$((x_{nm}x_{1m}^{-1}), \mathbf{e}) \in L$ for each $n \in N$. Since $L$ satisfies $(*)$, there exists a cross-sequence $(x_{nm(n)}x_{1m(n)}^{-1}) \in C[X]$ in $(x_{nm}x_{1m}^{-1})$ such that

(1) $((x_{nm(n)}x_{1m(n)}^{-1}), \mathbf{e}) \in L$,

(2) $m(n) \geqq n$ for all $n \in N$ and

(3) $((x_{nk(n)}x_{1k(n)}^{-1}), \mathbf{e}) \in L$ if $k(n) \geqq m(n)$ for all $n \in N$.

Since $[(x_{1m})] = \alpha$ and $L^*$ satisfies (SC 2), $[(x_{1m(n)})] = \alpha$, and hence $[(x_{nm(n)})] = \alpha$ by Lemma 4.4(4). Thus, we have a cross-sequence $(x_{nm(n)})$ in $(x_{nm})$ such that $(1)[(x_{nm(n)})] = \alpha$, and this cross-sequence satisfies also the properties (2) and (3).

Now we prove this Lemma in general case. Since $((\alpha_n), \alpha) \in L^*$, by definition of $L^*$, there exists $(y_n) \in C[X]$ such that $\alpha_m y_m^{-1} = \alpha[(y_n)]^{-1}$ for each $m \in N$. Hence we have a double sequence $(x_{nm}y_n^{-1})$ in $X$ with $[(x_{pm}y_p^{-1})] = \alpha[(y_n)]^{-1}$ for each $p \in N$. By above particular case, there exists a cross-sequence $(x_{n(m)}y_n^{-1}) \in C[X]$ such that $(1)[(x_{nm(n)}y_n^{-1})] = \alpha[(y_n)]^{-1}$, $(2) m(n) \geqq n$ for all $n \in N$ and $(3)[(x_{nk(n)}y_n^{-1})] = \alpha[(y_n)]^{-1}$ if $k(n) \geqq m(n)$ for all $n \in N$. Hence, by Lemma 4.4(4), we have a cross-sequence $(x_{nm(n)})$ in $(x_{nm})$ such that $(1)[(x_{nm(n)})] = \alpha$, and this cross-sequence satisfies also the properties (2) and (3).

Let $(X, L)$ be a sequential convergence space satisfying $(*)$ and let $A$ be any non-empty subset of $X$ and $L_A = \{((x_n), x) \in L | ((x_n), x) \in S[A] \times A\}$. Then, it is easy to verify that $(A, L_A)$ is a sequential convergence space satisfying $(*)$. In the usual sense, $(A, L_A)$ is called *a dense subspace of* $(X, L)$ if for each $x \in X - A$, there exists $(x_n) \in S[A]$ such that $((x_n), x) \in L$.

THEOREM 4.6. *Assume that $L^*$ satisfies the following condition:*

*$(**)$ Let $\alpha \in X^*$ and let $(\alpha_{nm})$ be a double sequence in $X^*$ with $((\alpha_{nm}), \alpha) \in L^*$ for each $n \in N$. It is possible to choose $(x_n) \in C[X]$ satisfying the property that for each $p \in N$, there exists a subsequence $(x_{n_p(m)})$ of $(x_n)$ such that $\alpha_{pm}x_{n_p(m)}^{-1} = \alpha[(x_n)]^{-1}$ for all $m \in N$. Then, $(X^*, *, L^*)$ is a complete sequential convergence group containing $(X, \cdot, L)$ as a dense subgroup.*

*Proof.* First, we shall show that $L^*$ satisfies $(*)$. Let $((\alpha_n), \alpha) \in L^*$ and let $(\alpha_{nm})$ be a double sequence in $X^*$ with $((\alpha_{nm}), \alpha_n) \in L^*$ for each $n \in N$. Since $((\alpha_n), \alpha) \in L^*$, there exists $(x_n) \in C[X]$ such that

$\alpha_m x_m^{-1} = \alpha[(x_n)]^{-1}$ for each $m \in N$, let $\alpha[(x_n)]^{-1} = \beta$. Then, by (**), there exists $(y_n) \in C[X]$ satisfying the property that for each $p \in N$, there exists a subsequence $(y_{n_p(m)})$ of $(y_n)$ such that $\alpha_{pm} x_p^{-1} y_{n_p(m)}^{-1} = \beta[(y_n)]^{-1}$ for all $m \in N$. Since $(y_n) \in C[X]$, by Lemma 4.4(2 and 3), we see that $[(y_{n_p(m)})] = [(y_n)]$ for each $p \in N$, and hence,by Lemma 4.5, there exists a cross-sequence $(y_{n_m(i(m)}) \in C[X]$ in the double sequence $(y_{n_p(m)})$ such that $(1)[(y_{n_m(i(m))})] = [(y_n)]$, $(2) i(m) \geqq m$ for all $m \in N$ and $(3)[(y_{n_m(j(m))})] = [(y_n)]$ if $j(m) \geqq i(m)$ for all $m \in N$. Thus we have that

$$((\alpha_{mi(m)} x_m^{-1} y_{n_m(i(m))}^{-1}), \beta[(y_n)]^{-1}) \in L^*,$$

i.e.,

$$\alpha_{1i(1)} x_1^{-1} y_{n1(i(1))}^{-1}, \alpha_{2i(2)} x_2^{-1} y_{n_2(i(2))}, \dots \to \beta[(y_n)]^{-1}.$$

Hence, by Lemma 4.4(5), $((\alpha_{ni(n)}), \alpha) \in L^*$. Since the cross-sequence $(y_{n_m(i(m))})$ satisfies the above properties (1), (2) and (3), it is easy to verify that $i(n) \geqq n$ for all $n \in N$ and $((\alpha_{nj(n)}), \alpha) \in L^*$ if $j(n) \geqq i(n)$ for all $n \in N$. Therefore, $L^*$ satisfies $(*)$.

Consequently, by Theorem 4.3 and Lemma 4.4, we have that $(X^*, *, L^*)$ is a sequential convergence group containing $(X, \cdot, L)$ as a dense subgroup.

Finally, we shall show that $(X^*, *, L^*)$ is complete. Let $(\alpha_n)$ be a Cauchy sequence in $X^*$. Then, by definition of Cauchy, $((\alpha_{i_n} \alpha_{j_n}^{-1}), \mathbf{e}) \in L^*$ for each subsequences $(\alpha_{i_n})$ and $(\alpha_{j_n})$ of $(\alpha_n)$. We divide this proof into two cases.

Case 1. There exists a subsequence $(\alpha_{i_n})$ of $(\alpha_n)$ such that $\alpha_{i_n} \alpha_{i_m}^{-1} \in X$ for each $n, m \in N$. Then, it is clear that $(\alpha_{i_n} \alpha_{i_1}^{-1}) \in C[X]$, and so $((\alpha_{i_n} \alpha_{i_1}^{-1}), [(\alpha_{i_n} \alpha_{i_1}^{-1})]) \in L^*$ by Lemma 4.4(3). Therefore, by Lemma 4.4(5), we have that $((\alpha_n), [(\alpha_{i_n})]) \in L^*$.

Case 2. There doesn't exist a subsequence $(\alpha_{i_n})$ of $(\alpha_n)$ such that $\alpha_{i_n} \alpha_{i_m}^{-1} \in X$ for each $n, m \in N$. Without loss of generality, we assume that $\alpha_n \alpha_m^{-1} \notin X$ for each $n \neq m \in N$. Now we construct a subsequence $(\alpha_{i_n})$ of $(\alpha_n)$. Let $\alpha_{i_1} = \alpha_2$. Then we can choose $\alpha_{i_2}$ in $\{\alpha_3, \alpha_4\}$ satisfying

$$(\alpha_1 \alpha_{i_1}^{-1})(\alpha_2 \alpha_{i_2}^{-1})^{-1} \notin X.$$

Assume that we have just chosen $k-1$ natural numbers $i_m$ such that

$$2^{m-1} \lneqq i_m \leqq 2^m$$

and that

$$(\alpha_n \alpha_{i_n}^{-1})(\alpha_m \alpha_{i_m}^{-1})^{-1} \notin X$$

for each $n \neq m \in \{1, 2, \ldots, k-1\}$. We will show that there exists $\alpha_{i_k}$ in $\{\alpha_{2^{k-1}+1}, \alpha_{2^{k-1}+2}, \ldots, \alpha_{2^k}\}$ such that

$$(\alpha_n \alpha_{i_n}^{-1})(\alpha_k \alpha_{i_k}^{-1})^{-1} \notin X$$

for each $n \in \{1, 2, \ldots, k-1\}$. Suppose that there doesn't exist such $\alpha_{i_k}$. Then

$$(\alpha_n \alpha_{i_n}^{-1})(\alpha_k \alpha_p^{-1})^{-1}) \in X$$

for all $p \in \{2^{k-1}+1, 2^{k-1}+2, \ldots, 2^k\}$ and for all $n \in \{1, 2, \ldots, k-1\}$, and hence we have that

$$\{(\alpha_n \alpha_{i_n}^{-1})(\alpha_k \alpha_p^{-1})^{-1}\}\{(\alpha_n \alpha_{i_n}^{-1})(\alpha_k \alpha_q^{-1})^{-1}\}^{-1} = \alpha_p \alpha_q \in X,$$

which is a contradiction. Thus, by induction, we obtain a subsequence $(\alpha_{i_n})$ of $(\alpha_n)$ such that $(\alpha_n \alpha_{i_n}^{-1})(\alpha_m \alpha_{i_m}^{-1})^{-1} \notin X$ for all $n \neq m \in N$. Since $(\alpha_n)$ is Cauchy in $X^*$, $((\alpha_n \alpha_{i_n}^{-1}), \mathrm{e}) \in L^*$, and therefore, by Lemma 4.4(4), $(\alpha_n \alpha_{i_n}^{-1})(\alpha_m \alpha_{i_m}^{-1})^{-1} \in X$ for each $n, m \in N$. This contradicts. It follows that the case 2 cannot occur. The proof completes.

*Comment.* The idea of proof of case 2 can be found in J. Novǎk[5].

*Example 4.7.* Let $L_1 = \{((x_n), x) \in S[Q] \times Q | (x_n)$ converges to $x$ in $Q$ with the usual topology$\}$ and $L_2 = \{((x_n), x) \in S[R] \times R | (x_n)$ converges to $x$ in $R$ with the usual topology$\}$, and let $+$ be the usual addition on $R$. Then, it is clear that $(Q, +, L_1)$ is a sequential convergence group and $L_1^*$ satisfies $(**)$. Hence $(Q, +, L_1)$ has two different completions: $(Q^*, +, L_1^*)$ and the usual completion $(R, +, L_2)$. Notice that $L_1^* \subsetneqq L_2$.

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