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A GENERALIZATION OF LICHNEROWICZ'S THEOREM

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1. Introduction

Let M be a compact Riemannian n-manifold and let λ_1 be the first nonzero eigenvalue of the Laplace operator acting on the space of C^∞ functions on M. Then Lichnerowicz has proved the following [Lic]: *If the Ricci curvature satisfies* $\operatorname{Ric} \geq (n-1)k$ *for some constant* $k \in \mathbb{R}$, *then* $\lambda_1 \geq nk$. In this paper we prove the following generalization.

THEOREM. *Let* $E \to M$ *be a flat Riemannian vector bundle and let* λ_1 *be the first nonzero eigenvalue of the Laplace operator acting on the space of smooth sections of* E. *If* $\operatorname{Ric}^E \geq (n-1)k$ *for some* $k \in \mathbb{R}$, *then* $\lambda_1 \geq nk$.

We will soon describe the meaning of the *Ricci curvature* Ric^E for the vector bundle E, which is equal to the ordinary Ricci curvature when E is the trivial line bundle. Clearly the above theorem generalizes the theorem of Lichnerowicz. The precise condition will be explained in 4.1.

The eigenvalues of the Laplacian Δ of a flat connection D are important to understand the *heat trace* $Z(t) = \sum e^{-\lambda t}$, or the *zeta function* $\zeta(s) = \sum_{\lambda \neq 0} \lambda^{-s}$, where λ runs through the spectrum of the Laplacian. These study are related to the index problem [APS] and analytic torsion [Fay, BZ].

For the proof, we use the Weitzenböck formula [Wu, Bou], which is briefly reviewed in section 2. This technique is often used to prove various vanishing theorems (2.2). It is also used in the study of gauge theory [BL, FU].

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2. Weitzenböck formula

Let $E \to M$ be a Riemannian vector bundle with a compatible Riemannian connection D over a compact Riemannian manifold M. Then we have the induced exterior derivative

$$d_D : A^p(E) \to A^{p+1}(E),$$

where $A^p(E) = A^0(\wedge^p TM^* \otimes E)$ denotes the space of smooth p-forms on M with values in E. Let d_D^* denote the formal adjoint of d_D and let $\Delta = d_D d_D^* + d_D^* d_D$ be the Laplacian. We denote the *covariant derivative* by

$$\nabla : A^0(\wedge^p TM^* \otimes E) \to A^0(TM^* \otimes \wedge^p TM^* \otimes E)$$

and its formal adjoint by ∇^* so that $\nabla^*\nabla$ is the *rough Laplacian.* An element $\xi \in A^p(E)$ is said to be *harmonic* if $\Delta\xi = 0$ and said to be *parallel* if $\nabla\xi = 0$. Since the anti-symmetrization of $\nabla\xi$ is equal to $d_D\xi$, a parallel section $s \in A^0(E)$ is harmonic.

We also define a vector bundle endomorphism

$$\mathcal{R}^p : \wedge^p TM^* \otimes E \to \wedge^p TM^* \otimes E$$

as follows: Let $v_1, \ldots, v_n$ be an orthonormal basis for the tangent space TM_m of M at a point $m \in M$, and let $\theta^1, \ldots, \theta^n \in TM_m^*$ denote the dual basis. Then

$$\mathcal{R}^p(\xi) := -\sum_{i,j=1}^{n} \theta^i \wedge \operatorname{int}(v_j) R^p_{v_i v_j}(\xi), \qquad \forall \xi \in (\wedge^p TM \otimes E)_m,$$

where R^p denotes the curvature tensor for the bundle $\wedge^p TM^* \otimes E$, and $\operatorname{int}(v) : \wedge^p TM^* \otimes E \to \wedge^{p-1} TM^* \otimes E$ denotes the interior product. The operator $\mathcal{R}^p$ is well-defined, i.e., independent of the choice of orthonormal basis $v_1, \ldots, v_n$. It is easy to see that $\mathcal{R}^p$ is self-adjoint. When E is the trivial line bundle and $p = 1$, we have

$$\mathcal{R}^1 = \operatorname{Ric} : TM^* \to TM^*,$$

the (dual of the) ordinary Ricci curvature of M.

Now the *Weitzenböck formula* says that

$$\Delta = \nabla^*\nabla + \mathcal{R}^p \qquad \text{on } A^p(E). \tag{2.1}$$

This is an easy consequence of the following identities:

$$d_D\xi(m) = \sum_i \theta^i \wedge \nabla_{v_i}\xi$$
$$d_D^*\xi(m) = -\sum_i \mathrm{int}(v_i)\nabla_{v_i}\xi$$

for any $\xi \in A^p(E)$.

For sections $\xi_1, \xi_2 \in A^p(E)$, we have a pointwise inner product $\langle \xi_1, \xi_2 \rangle$ and its total integral

$$\langle\langle \xi_1, \xi_2 \rangle\rangle := \int_M \langle \xi_1, \xi_2 \rangle \, \delta g$$

where δg denotes the Riemannian density of M.

The Weitzenböck formula is often used to prove vanishing theorems, e.g.,

COROLLARY 2.2. *Suppose $\langle\langle \mathcal{R}^p\xi, \xi \rangle\rangle \geq 0$ for any $\xi \in A^p(E)$. Then the dimension of the space of harmonic sections of $\wedge^p TM^* \otimes E$ is less than or equal to $r\binom{n}{p}$, where r is the rank of E. If $\langle\langle \mathcal{R}^p\xi, \xi \rangle\rangle > 0$ for any nonzero $\xi \in A^p(E)$, then there are no nontrivial harmonic sections in $A^p(E)$.*

Proof. Note that if $\xi \in A^p(E)$, then the Weitzenböck formula (2.1) implies, after the integration, that

$$\langle\langle \Delta\xi, \xi \rangle\rangle = \|\nabla\xi\|^2 + \langle\langle \mathcal{R}^p\xi, \xi \rangle\rangle.$$

Thus if ξ is harmonic, then

$$0 = \|\nabla\xi\|^2 + \langle\langle \mathcal{R}^p\xi, \xi \rangle\rangle$$

Thus the condition implies that ξ is parallel and hence it is determined by its value at a point. Now the conclusion is trivial.

3. Hessian

For a section s of $E \to M$, the *Hessian* of s is a bilinear bundle homomorphism

$$\operatorname{Hess} s : TM \times TM \to E$$

defined by

$$(\operatorname{Hess} s)(V, W) = \nabla^2_{VW} s,$$

where V and W are vector fields on M and $\nabla^2_{VW} s := \nabla_V \nabla_W s - \nabla_{\nabla_V W} s$. Note that

$$(\operatorname{Hess} s)(V, W) - (\operatorname{Hess} s)(W, V) = R^E_{VW} s.$$

In particular, $\operatorname{Hess} s$ is symmetric if and only if D is flat.

LEMMA 3.1. $|\operatorname{Hess} s|^2 \geq \frac{1}{n}|\Delta s|^2$ *for any* $s \in A^0(E)$.

Proof. Fix a point $m \in M$ and an orthonormal frame field $V_1, \ldots, V_n$ for the tangent bundle TM of M around m such that $\nabla V_i(m) = 0$ for all $i = 1, \ldots, n$. Then

$$\begin{aligned} |\operatorname{Hess} s|^2(m) &= \sum_{i,j} |\nabla^2_{V_i V_j} s(m)|^2 = \sum_{i,j} |\nabla_{V_i} \nabla_{V_j} s(m)|^2 \\ &\geq \sum_i |\nabla_{V_i} \nabla_{V_i} s(m)|^2 \geq \frac{1}{n} \left(\sum_i |\nabla_{V_i} \nabla_{V_i} s(m)| \right)^2 \\ &\geq \frac{1}{n} \left| \sum_i \nabla_{V_i} \nabla_{V_i} s(m) \right|^2 = \frac{1}{n} |\Delta s|^2(m). \end{aligned}$$

This completes the proof.

4. Proof of the Theorem

We now assume that D is flat and

$$(4.1) \qquad \langle\langle \mathcal{R}^1 \xi, \xi \rangle\rangle \geq (n-1) k \|\xi\|^2, \qquad \forall \xi \in d_D(A^0(E)) \subset A^1(E).$$

Let $s \in A^0(E)$ be a nonzero λ_1-eigensection for the Laplacian Δ. Then by the Weitzenböck formula (2.1),

$$\begin{aligned}
\langle\!\langle \Delta d_D s, d_D s\rangle\!\rangle &= \langle\!\langle \nabla^*\nabla d_D s, d_D s\rangle\!\rangle + \langle\!\langle \mathcal{R}^1 d_D s, d_D s\rangle\!\rangle \\
&\geq \|\nabla d_D s\|^2 + (n-1)k\|d_D s\|^2 \\
&= \|\operatorname{Hess} s\|^2 + (n-1)k\|d_D s\|^2 \\
&\geq \frac{1}{n}\|\Delta s\|^2 + (n-1)k\|d_D s\|^2 \qquad \text{by (3.1)} \\
&= \frac{\lambda_1}{n}\|d_D s\|^2 + (n-1)k\|d_D s\|^2 \quad \text{since } \Delta s = \lambda_1 s.
\end{aligned}$$

Since D is flat, Δ commutes with d_D and hence the left hand side is equal to

$$\langle\!\langle \Delta d_D s, d_D s\rangle\!\rangle = \langle\!\langle d_D \Delta s, d_D s\rangle\!\rangle = \lambda_1 \|d_D s\|^2.$$

Thus we have

$$\lambda_1(1 - \frac{1}{n})\|d_D s\|^2 \geq (n-1)k\|d_D s\|^2.$$

Since $\lambda_1 \neq 0$, we have $d_D s \neq 0$ and hence $\lambda_1 \geq nk$. This completes the proof.

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