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A NOTE ON ASYMPTOTIC GAUSS-BONNET THEOREMS

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1. Introduction

The total curvature on a Riemannian surface M is defined to be the improper integral over M of Gaussian curvature G :

$$c(M) := \int_M G\, dV_M,$$

where dV_M is the volume form of M. In [10, 11], Shiohama and Shioya studied the behavior of geodesic rays on ends of noncompact, complete, finitely connected, Riemannian surfaces which admit total curvature and they found a relation between the amount of rays and the total curvature.

If M admits total curvature, i.e. the total curvature integral converges, the Cohn-Vossen inequality[2] says that

$$c(M) \leq 2\pi\chi(M),$$

where $\chi(M)$ is the Euler characteristic of M. Obviously the total curvature depends on the Riemannian metric involved and, therefore, is not a topological invariant. There arises the question if one can explain geometrically the meaning of the total curvature. This is from the philosophy that the difference between $2\pi\chi(M)$ and the total curvature must tell us about the asymptotic geometry at the opening of the manifold at the infinity.

Following the work of Maeda[5] and Shiga[6, 7], Shiohama found several geometric properties of complete surfaces admitting total curvatures.[8, 9, 10] His work depends essentially on the earlier investigations of Fiala[3] and Hartman[4]. Fiala described in several ways how to fill

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the gap in the Cohn-Vossen inequality and generate an equality out of it. His work depends on explicit formula on the derivative of the length of geodesic circles and is done in the setting of analytic metrics. Hartman described the behavior of the geodesics orthogonal to a given curve and generalized Fiala's work to smooth metrics.

Shiohama generalized Fiala's and Hartman's results and explained the relationships between total curvature and geometric behavior at the ends of the manifold which is given below. A ray is, by definition, a geodesic defined on $[0,\infty)$ such that it is distance minimizing between any pair of points on it. Let $p \in M$ and define A_p be the set of unit tangent vectors at p which is tangent to rays emanating from p. We denote by μ the standard measure on the unit circle in the tangent space $\mathbf{T}_pM$.

THEOREM 1. (Shiohama[10]) *Assume that a Riemannian surface with one end admits total curvature $c(M)$ with $2\pi\chi(M) = c(M) < 2\pi$. Let p_i be an arbitrary divergent sequence of points in M. Then*

$$\lim_{i\to\infty} \mu(A_{p_i}) = 2\pi\chi(M) - c(M).$$

This theorem is particularly interesting because it shows geometrically what happens at the ends of a complete manifold. The situation is considered analogous to a compact manifold from which a disk is cut off in which case the total curvature is related to the total geodesic curvature of the boundary by the Gauss-Bonnet formula. Therefore such a formula on complete Riemannian surfaces can be considered as a generalized Gauss-Bonnet formula. The consideration of the set of rays is closely related to the compactification by Anderson and Schoen of the manifolds with bounded negative curvature [1]. The geometry of ideal boundary is being studied by some of the mathematicians.(Cf. [12]) Following Shiohama, Shioya generalized the result to the case when $2\pi\chi(M) - c(M) \geq 2\pi$ and found that the limit equals 2π.[11]

When M is multiply connected the proof of the theorem involves some technical difficulties, but the essential argument lies in the case when M is simply connected. Also the essential features of the theorem is easily seen under this assumption. In this paper we prove both cases at the same time and show the essential feature of the theorem. This will also simplify some of their arguments. For some of the lemmas, just the sketches of the proofs will be given. From now on we assume that the topology of M is trivial so that M is diffeomorphic to $\mathbb{R}^2$.

2. Definitions

For geometric definitions and the facts we state in this section, we mostly follow Shiohama[8] and Shioya[11], which are referred to for detailed investigations of the geometry of radial geodesics and cut loci. Let M be as defined above. We fix a compact connected set K in M so that the total curvature $c(K)$ of K is almost that of M, i.e. $|c(K)-c(M)| < \epsilon$ for $\epsilon > 0$ sufficiently small. Take a point p outside K. This point will eventually diverge to infinity and we may assume that it is sufficiently far away. Now take a real number $R > 0$ and consider the closed geodesic ball $\mathbf{B}(p, R)$, where R is chosen so large that $\mathbf{B}(p, R)$ contains K. Let $A(p)$ be the set of rays emanating from p. Here we may either consider each ray as the set of points on it, or associate a unit tangent vector to the ray in $\mathbf{T}_pM$ and consider $A(p)$ as a subset of the unit circle $S(p)$ in the tangent space. Also we denote by μ the Lebesgue measure on the unit circle. This way we can talk about the measure of rays or geodesics emanating from p. $A_p(K)$ denotes the set of rays in $A(p)$ intersecting K, and $A'_p(K)$ denotes the closure of the set of those in $A(p)$ which do not intersect the interior of K. Since a limit of rays is a ray again, $A(p)$ is closed and $S(p) \setminus A(p)$ consists of (at most) countably many disjoint open intervals of rays.

For each point $q \in \partial K$ there is a minimizing geodesic joining p and q. All such geodesic directions form a compact subset of the unit circle in T_pM, of which the measure is denoted by $\theta_p(K)$. Let $\theta_p(K)$ be the inner angle of D_K at p.

3. Asymptotic Gauss-Bonnet formula

The most interesting case for our purpose is when the total curvature converges to a finite limit so that outside a compact set the total curvature is almost zero. But we do not assume that the curvature is bounded uniformly or by any radial function. In our discussion of Theorem 1, we will generalize some of their results so that one can argue both cases at the same time.

The most important result of all that will be used in this paper is the geometry of the boundary curve of the geodesic ball and the cut locus of a given curve which goes back to Fiala[3] and Hartman[4] and later was improved by Shiohama.[8] For these we state only very briefly and refer the readers to the references above. For our purpose we only need to consider the geodesic balls and the cut loci of a point p. In our case of surfaces with total curvature, the boundary curves of geodesic balls are all homeomorphic to a circle for sufficiently large radii, and are piecewise smooth for almost every radii, i.e. except for a measure-zero set of radii. The cut locus consists of several continuous curves diverging to infinity which are disjoint from each other. The minimizing geodesics from a cut point to p bounds a disk domain which is monotone increasing as the cut point moves along the cut locus. The sum of inner angles between the pairs of geodesics at all the cut point of distance R of such domains approaches 0 as R approaches infinity.

On $\mathbf{B}(p, R)$, consider all the minimizing geodesics joining p and the boundary points of the ball which does not meet K. Denote by $D'_K(R)$ the closure of union of all the disk domains disjoint from K and bounded by two such geodesics and the boundary of the ball. Let $\theta'_p(K, R)$ the inner angle of $D'_K(R)$ at p and let

$$\theta'_p(K) = \lim_{R \to \infty} \theta'_p(K, R).$$

LEMMA 1. (1) $\theta_p(K) \to 0$ *as* p *approaches infinity.*
(2) $\theta'_p(K, R) \to \mu(A'_p(K))$ *as* $R \to \infty$.
(3) $\mu(A'_p(K)) - \mu(A(p)) \to 0$ *as* p *approaches infinity.*

(1) holds for arbitrary compact K. It says that the viewing angle of a compact set from a point gets arbitrarily small as one walks away to infinity.

The proof of (1) of the Lemma 1 is a repeated application of arguments in the proof of Theorem A and C of Shiohama in [8]. (2) of the Lemma 1 holds because $A'_p(K) \subset D'_K(R)$ and the radial boundary curves of $D'_K(R)$ converge to those of $A'_p(K)$ as R approaches infinity, and also because this convergence is uniform in any compact sets.(There may be countably infinite number of radial geodesic boundary curves of $A'_p(K)$.) (3) is an immediate consequence of (1).

LEMMA 2 (Asymptotic Gauss-Bonnet formula of Fiala-Hartman[3,4]).

If we denote by $L_p(R)$ the length of the boundary curve of $\mathbf{B}(p,R)$ then

$$\lim_{R\to\infty} \frac{L_p(R)}{R} = 2\pi - c(M).$$

Using this lemma and the fact that the subdomain of $\mathbf{B}(p,R) \setminus A(p)$ covered by minimizing geodesics from p to the boundary of the ball shrinks in its width uniformly on compact sets, we easily get the following

COROLLARY. *Let $C'(p,R)$ be the boundary of $\mathbf{B}(p,R)$ not lying in $A(p)$ and $L(C'(p,R))$ the length of $C'(p,R)$, then as R approaches ∞*

(1) $\lim L(C'(p,R))/R = 0$, *and*
(2) $\lim \int_{C'(p,R)} \kappa_g = 0$.

Therefore the role of the boundary of the ball, in the asymptotic geometry of the domain $D'_K(R)$, is only important on the portion lying in the set $A_p(K)$.

LEMMA 3 (Shiohama[10]). *Let D be a domain bounded by two rays emanating from a point p and assume that there is no other ray from p in it. Then the total curvature of the domain equals the inner angle between the two rays at p.*

The following lemma is due to Cohn-Vossen and found in Shioya[11].

LEMMA 4. *Let D be a domain bounded by piecewise smooth curves and the boundary curve c is homeomorphic to $\mathbb{R}$. If $c|(-\infty,a]$, $c|[b,\infty)$ are geodesics for some $a,b \in \mathbb{R}$, and if $d_D(c(t),c(-t)) \geq 2t - r$ for all $t \geq 0$ and for some constant r, then*

$$c(D) \leq 2\pi\chi(D) - 2\pi - \kappa(D),$$

where d_D is the induced inner distance on the closure of D and $\kappa(D)$ is the total geodesic curvature of D.

Using the standard Gauss-Bonnet formula, one deduces the following

THEOREM A. (1) *If there is no ray from p through K,*

$$\lim_{p\to\infty} \lim_{R\to\infty} [2\pi - \theta'_p(K,R)] = c(M).$$

(2) *If there is a ray from p_i through K for a sequence $p_i \to \infty$, then there is a line through K and*

$$\lim_{p\to\infty} \lim_{R\to\infty} [2\pi - \theta'_p(K, R)] = 0.$$

Proof. (1) is obvious from Lemma 1(1) and the Gauss-Bonnet formula. For (2), first we observe the following. For a ray γ_i from p_i through K, choose a point q_i on $\gamma_i \cap K$. Since the points q_i lies inside a compact set the direction vectors at q_i of γ_i's have a convergent subsequence in the tangent bundle. The limit is a geodesic line ℓ passing through K. Without loss of generality, we may consider only those γ_i's converging to the line. Now assume that $2\pi - \theta'_{p_i}(K)$ stays $\geq \alpha > 0$. From Lemma 1 (1), the viewing angle from p_i of K is negligible compared to α. Therefore there is an open disk domain E_i for each i, whose boundary consists of one of the rays which bounds $D'_K(R)$, a ray which bounds $A_{p_i}(K)$ so that the inner angle ψ_i at p_i is still staying away from 0 and there is no other ray from p_i inside the domain. Moreover, $\cup_i E_i = H$ is the half plane bounded by ℓ. Since $\psi_i = \kappa(E_i)$, we have

$$0 < \lim \psi_i = \lim c(E_i) = c(H) \leq -\kappa(H) = 0,$$

which is a contradiction. This completes the proof.

This theorem together with the lemmas above gives the results of Shiohama and Shioya. When $c(M) \geq 0$, from Lemma 1 and 3 we get $|2\pi - \mu(A(p)) - c(K)| < \epsilon$. As we increase K to exhaust M and move p to infinity accordingly, we see that

$$2\pi - c(M) = \lim \mu(A(p)).$$

When $-\infty < c(M) < 0$, same argument as above works if there is no geodesic line through K. If there is one, then again Theorem A(2) gives the desired result;

$$2\pi = \lim \mu(A(p)).$$

Now to understand the geometry of the surfaces with finite total curvature, it is instructive to observe the following simple fact. We look at the domains consisting of minimizing geodesics joining p and boundary points of $\mathbf{B}(p;R)$ which lie inside $D'_K(R)$, We consider the closed set of radial geodesics which form a smooth normal variation. This is possible because of the fact that for almost all sufficiently large radius the boundary circle is a piecewise smooth curve. Now consider the geodesic normal coordinate on those geodesic variations. The Riemannian metric on these geodesics are given by

$$dr^2 + f^2\, d\theta^2,$$

where r is the distance from p and θ is the angle at p. Then $\partial/\partial\theta$ being a Jacobi field, the Gaussian curvature function G satisfies the equation

$$f'' + G\, f = 0,$$

where the differentiation is with respect to r. Therefore on a sector S covered by such a variation, the total curvature formula is

$$\begin{aligned}\int G &= \int_{[0,R]} \int -\frac{f''}{f} \cdot f\, d\theta\, dr = \int_{[0,R]} \int -f'' d\theta\, dr \\ &= \int_{[0,R]} -L'' dr = L'_S(0^+) - L'_S(R),\end{aligned}$$

where $L_S(r)$ is the length of the arc of the circle of radius r in S. Thus the existence of a finite total curvature means the existence of the limit of $L'_S(R)$ as $R \to \infty$. And hence the existence of total curvature suggests the similarity of such surfaces to that of a Euclidean cone. This is an infinitesimal version of Fiala and Hartman. From this viewpoint, in the case of negative total curvature, the measure of rays considered by Shioya is not showing the asymptotic geometry in the part of the rays through the set of concentration of the curvature. This we hope to investigate in a forthcoming paper.

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