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GEOMETRIC INVARIANTS FOR LIAISON OF SPACE CURVES LYING ON A SMOOTH CUBIC SURFACE

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0. Introduction

Let k be an algebraically closed field and let $S = k[x_0, x_1, x_2, x_3]$. By a curve we mean a closed, one-dimensional subscheme of $\mathbf{P}^3$ which is equidimensional and locally Cohen-Macaulay. We say that two curves C and C' in $\mathbf{P}^3$ are directly linked by a complete intersection X of two surfaces, written $C \sim_X C'$, if

(1) C, C' have no component in common,

(2) $C \cup C' = X$ scheme theoretically (i.e., $I_C \cap I_{C'} = I_X$).

C is linked (resp. evenly linked, oddly linked) to C' if C' can be obtained from C by a finite (resp. even, odd) succession of direct links. We then write $C \sim C'$ (resp. $C \sim_e C'$, $C \sim_o C'$). The equivalence relation generated by direct linkages is called liaison. It was shown in [LR] that for a general smooth irreducible curve $C \subsetneq \mathbf{P}^3$ of sufficiently large degree, if C' is a curve linked to C, other than C itself, then $\deg(C') > \deg(C)$ and $P_a(C') > P_a(C)$. Accordingly if C and C' are curves with the same degree and genus, then they are not linked. In this note we study what the geometric invariants are if C is linked to C' of the same degree d and genus g when $g = 2,\ d = 6 :\ g = 3,\ d = 7$. And these invariants will narrow down the possibilities for C to be linked to C' with the same degree d and genus g. Hence these curves will be examples to demonstrate that results in [LR] are also true for curves with small degree.

1. Preliminaries

The main result concerning liaison equivalence classes of curves involves the Hartshorne-Rao module $M(C) = \oplus_{n \in \mathbf{Z}} H^1(\mathbf{P}^3, I_C(n))$:

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THEOREM 1.1. (a) (Hartshorne) *If* $C \sim_X C'$ *where* $I(X) = (F_1, F_2)$ *and* $degF_i = d_i$ $(i = 1, 2)$, *then* $M(C) \cong M'(C')(4 - d_1 - d_2)$, *where* $M'(C') = Hom_k(M(C'), k)$.

(b) (Rao) *If* $M(C) \cong M(C')(\nu)$ *for some* $\nu \in \mathbf{Z}$, *then* $C \sim C'$.

(c) (Rao) *If* M *is any graded* S*-module of finite length, then there exists a smooth curve* C *such that* $M(C) \cong M(\nu)$ *for some* $\nu \in \mathbf{Z}$.

Proof. See [R].

Let $M = \oplus_{n \in \mathbf{Z}} M_n$ be a graded S-module of finite length, and let $S_1 = H^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(1))$. The S-module structure of M is given by the collection of vector space homomorphisms $\phi_n : S_1 \to \mathrm{Hom}_k(M_n, M_{n+1})$. If we choose bases for M_n and M_{n+1}, and if $L = a_0X_0 + a_1X_1 + a_2X_2 + a_3X_3 \in S_1$, then ϕ_n can be viewed as a $(\dim M_{n+1}) \times (\dim M_n)$ matrix A_n whose entries are linear polynomials in the a_i.

DEFINITION 1.2. Let $1 \le r + 1 \le \min\{\dim M_n, \dim M_{n+1}\}$. Then $W_{n,r}$ is the closed subscheme of $(\mathbf{P}^3)^*$ defined by all the $(r+1) \times (r+1)$ minors of A_n, and $V_{n,r}$ is the variety on which $W_{n,r}$ is supported. Equivalently, $V_{n,r} = \{ L^* \in (\mathbf{P}^3)^* \mid rk\phi_n(L) \le r \}$, and from this we extended the definition of $V_{n,r}$ to include all integers n and r.

Note that $V_{n,r} \subseteq V_{n,r+1}$ for all r, and $V_{n,r} = (\mathbf{P}^3)^*$ for $r \gg 0$ and we shall be primarily concerned with the last $V_{n,r}$ which is a proper subvariety of $(\mathbf{P}^3)^*$. On the other hand the varieties $V_{n,r}$ are independent of the choice of vector space bases for the M_n. Hence they are isomorphism invariants of the module M. Furthermore, since the transpose matrix ${}^t\phi_n(L)$ has the same $(r+1) \times (r+1)$ minors, it follows that the dual module $M' = \mathrm{Hom}_k(M, k)$ has the same collection of varieties $V_{n,r}$, but in the reverse order: $V'_{n,r} = V_{-n-1,r}$. Finally, it is clear that they are preserved under shifts of M or M'. Therefore $V_{n,r}$ are invariants of a given liaison class by Theorem 1.1.

The following fact, relating the degrees and arithmetic genera of linked curves, is often useful:

LEMMA 1.3. *Let* $C \sim_X C'$ *as above. Then*

(a) $degC' = d_1 d_2 - degC$.

(b) $P_a(C') - P_a(C) = \frac{1}{2}(d_1 + d_2 - 4)(degC' - degC)$.

Proof. See [M1] p. 550.

LEMMA 1.4. *Let $C \in \mathbf{P}^r$ $(r \geq 2)$ be an irreducible nondegenerate, possibly singular, curve of degree d. Then a general hyperplane meets C in d points any r of which are linearly independent.*

Proof. See [ACGH] p. 109.

LEMMA 1.5. *Let A be a $q \times p$ matrix of linear forms in $m+1$ variables, and let Y_r be the subscheme of $\mathbf{P}^m$ defined by the vanishing of the $(r+1) \times (r+1)$ minors of A. If $Y_r \neq \emptyset$ has the expected codimension $(p-r)(q-r)$, then*

$$deg Y_r = \prod_{i=0}^{p-r-1} [\binom{q+i}{r} / \binom{r+i}{r}].$$

Proof. See [M1] p. 550.

2. Main results

THEOREM 2.1. *Let C be an irreducible nondegenerate smooth curve of degree d in $\mathbf{P}^3$. Let $M(C)_l = H^1(\mathbf{P}^r, I_C(l))$. If $M(C)_l = 0$ for some l with $l \geq \frac{d-3}{2}$, then $M(C)_{l+1} = 0$.*

Proof. Let $C \cap H$ be a generic hyperplane section of C. Consider the following exact sequence

$$0 \longrightarrow I_C(l) \longrightarrow I_C(l+1) \longrightarrow I_{C\cap H}(l+1) \longrightarrow 0.$$

Taking cohomology, we get

(1)

$$0 \to H^0(\mathbf{P}^3, I_C(l)) \to H^0(\mathbf{P}^3, I_C(l+1)) \to H^0(\mathbf{P}^2, I_{C\cap H}(l+1)) \to 0$$

since we assume $M(C)_l = 0$. On the other hand, in the following exact sequence

$$0 \longrightarrow I_C(l) \longrightarrow \mathcal{O}_{\mathbf{P}^3}(l) \longrightarrow \mathcal{O}_C(l) \longrightarrow 0,$$

by taking cohomology, we also get

$$0 \longrightarrow H^0(\mathbf{P}^3, I_C(l)) \longrightarrow H^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(l)) \longrightarrow H^0(\mathbf{P}^3, \mathcal{O}_C(l)) \longrightarrow 0$$

since $M(C)_l = 0$. Therefore we have

$$h^0(\mathbf{P}^3, I_C(l)) = \frac{(l+3)(l+2)(l+1)}{6} - (dl - g + 1 + i)$$

where i is the index of specialty. Moreover, by general position theorem any three points in $C \cap H$ are linearly independent. Thus $C \cap H = p_1, \cdots, p_d$ impose independent conditions on the homogeneous polynomials of degree $l+1$ by Lemma 1.4 since $d \leq 2l+3$. Therefore

$$h^0(\mathbf{P}^2, I_{C\cap H}(l+1)) = \frac{(l+3)(l+2)}{2} - d$$

and hence we have from (1)

$$\begin{aligned} h^0(\mathbf{P}^3, I_C(l+1)) &= h^0(\mathbf{P}^3, I_C(l)) + h^0(\mathbf{P}^2, I_{C\cap H}(l+1)) \\ &= \frac{(l+4)(l+3)(l+2)}{6} - (dl - g + i + 1) - d. \end{aligned}$$

Now consider the following exact sequence

$$\begin{aligned} 0 \longrightarrow H^0(\mathbf{P}^3, I_C(l+1)) &\longrightarrow H^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(l+1)) \longrightarrow \\ &\longrightarrow H^0(C, \mathcal{O}_C(l+1)) \longrightarrow H^1(\mathbf{P}^3, I_C(l+1)) \longrightarrow 0. \end{aligned}$$

Because $h^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(l+1)) = \frac{(l+4)(l+3)(l+2)}{6}$ and $h^0(C, \mathcal{O}_C(l+1)) = d(l+1) - g + 1 + j$ where j is the index of specialty, we get $h^1(\mathbf{P}^3, I_C(l+1)) = -i + j$. But we also know that $i \geq j$ and $h^1(\mathbf{P}^3, I_C(l+1)) \geq 0$. Therefore $M(C)_{l+1} = 0$.

REMARK 2.2. In Theorem 2.1, $l \geq \frac{d-3}{2}$ means that $d \leq 2l+3$. One can see that the above bound "$d \leq 2l+3$" is sharp as in the following example.: Let C be a smooth irreducible curve of $d = 6$ and $g = 3$ on a smooth quadric hypersurface in $\mathbf{P}^3$, then $\dim M(C)_1 = 0$ but $\dim M(C)_2 = 1$. In this case, $d = 6 = 2 \cdot 1 + 4 = 2l + 4$.

THEOREM 2.3. *Let C and C' be smooth irreducible nondegenerate curves of genus $g = 2$ and degree $d = 6$.*

(a) *C (resp. C') lies on a cubic surface S (resp. S').*

(b) *If S (resp. S') is smooth, then C (resp. C') has a unique quadric-secant L_1 (resp. L_1') and a unique line L_2 (resp. L_2') which lies on S (resp. S') and disjoint from C (resp. C'). Moreover, L_1 (resp. L_1') meets L_2 (resp. L_2').*

(c) *If C and C' lie on a smooth cubic surface then $C \sim C'$ if and only if $L_1 \cap L_2 = L_1' \cap L_2'$.*

Proof. In the following exact sequence

$$0 \longrightarrow H^0(\mathbf{P}^3, I_C(l)) \longrightarrow H^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(l)) \longrightarrow$$
$$\longrightarrow H^0(C, \mathcal{O}_C(l)) \longrightarrow H^1(\mathbf{P}^3, I_C(l)) \longrightarrow 0,$$

we see that $h^1(\mathbf{P}^3, I_C(1)) = 1$, $h^1(\mathbf{P}^3, I_C(2)) = 1$ and $h^0(\mathbf{P}^3, I_C(3)) \geq 3$. Let S_1 and S_2 be the cubic surfaces containing C, then $S_1 \cap S_2 = C \cup D$ where $\deg D = 3$ and $P_a(D) = -1$ by Lemma 1.3. Therefore D is the disjoint union of a conic and a line. By simple calculation, we see that $\dim M(D)_0 = \dim M(D)_1 = 1$ and all other components are zero. And hence we get $\dim M(C)_1 = \dim M(C)_2 = 1$ and all other components are zero by Theorem 2.1.

On the other hand since C lies on a smooth cubic surface S, we have $C \sim al - \sum_{i=1}^{6} b_i e_i$ where l and e_i are generators of $PicS \cong \mathbf{Z}^7$. Then

$$(*) \qquad a > 0 \text{ and } b_i > 0 \quad \text{for each i,}$$
$$a > b_i + b_j \quad \text{for each i, j,}$$
$$2a > \sum_{i \neq j} b_j \quad \text{for each j}$$

because C is irreducible and smooth. Furthermore,

$$(1) \qquad \deg C = 3a - \sum b_i = 6,$$

$$(2) \qquad P_a(C) = \frac{1}{2}(a^2 - \sum b_i^2 - d) + 1 = 2.$$

Recall Schwarz's inequality, which says that if $x_1, x_2, \ldots, y_1, y_2 \ldots$ are two sequences of real numbers, then

$$|\sum x_i y_i|^2 \leq |\sum x_i^2| \cdot |\sum y_i^2|.$$

Taking $x_i = 1$, $y_i = b_i$, $i = 1, \cdots, 6$, we find $(\sum b_i)^2 \le 6(\sum b_i^2)$. Substitute $\sum b_i = 3a - 6$ and $\sum b_i^2 = a^2 - 8$ from (1) and (2), we obtain $a^2 - 12a + 28 \le 0$. Therefore we have $4 \le a \le 8$. We quickly find all possible values of the b_i satisfying $(*)$ by trial and we see that there are 30 linear systems of smooth sextics with $g = 2$ of type $(a : b_i) = (5 : 3, 2, 1, 1, 1, 1)$ and 90 linear systems of type $(a : b_i) = (6 : 3, 3, 2, 2, 1, 1)$. Moreover, we know that a smooth cubic surface contains 27 lines i.e., $E_i \sim e_i$, $F_{ij} \sim l - e_i - e_j$, $G_j \sim 2l - \sum_{i \ne j} e_i$. Consequently, we know that in both cases C has a unique quadricsecant L_1 and a unique line L_2 on S which is disjoint from C by calculating intersection number and using the facts that $l^2 = 1$, $e_i^2 = -1$, $l \cdot e_i = 0$ and $e_i \cdot e_j = 0$ for $i \ne j$. Also these two lines meet in both cases and hence (b) is proved.

To prove (c), we consider the following exact sequence

$$
\begin{aligned}
0 \longrightarrow H^0(\mathbf{P}^3, I_C(1)) \longrightarrow H^0(\mathbf{P}^3, I_C(2)) \longrightarrow \\
\longrightarrow H^0(\mathbf{P}^2, I_{C \cap H}(2)) \longrightarrow M(C)_1 \stackrel{\phi_1(L)}{\longrightarrow} M(C)_2 \longrightarrow
\end{aligned}
$$

where H is the hyperplane defined by $L = 0$. Then $L^* \in V_{1,0} = \{ L^* \in (\mathbf{P}^3)^* \,|\, rk\phi_1(L) \le 0 \}$ if and only if $h^0(\mathbf{P}^2, I_{C \cap H}(2)) = 1$ since $h^0(\mathbf{P}^3, I_C(2)) = 0$ and $\dim M(C)_1 = \dim M(C)_2 = 1$. This happens if and only if either four points of $C \cap H$ are collinear or $C \cap H$ lies on an irreducible conic. Any plane H through the quadricsecant L_1 of C meets C in two more points and hence the six points of $C \cap H$ lie on a reducible conic. Thus $L_1^* \in V_{1,0}$. Any plane H through the disjoint line L_2 from C meets S in the union of L_2 and a conic, so the six points of $C \cap H$ must be coconical. Hence $L_2^* \in V_{1,0}$.

Since C does not lie on a quadric hypersurface, not every hyperplane section of C lies on a conic. Accordingly $V_{1,0}$ has the excepted codimension $(1-0)(1-0) = 1$ and hence deg$V_{1,0} = 1$ by Lemma 1.5 i.e., $V_{1,0}$ is hyperplane. Therefore $V_{1,0}$ must be the plane in $(\mathbf{P}^3)^*$ which is dual to $L_1 \cap L_2$ by the above paragraph. Since $V_{1,0}$ is the invariant of a given liaison class, the conclusion follows.

THEOREM 2.4. *Let C be a smooth irreducible nondegenerate curve of degree $d = 7$ with genus $g = 3$ in $\mathbf{P}^3$. Then $\dim M(C)_1 = 1$, $\dim M(C)_2 = 2$, and all other components are zero.*

Proof. In the following exact sequence

$$0 \longrightarrow H^0(\mathbf{P}^3, I_C(l)) \longrightarrow H^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(l)) \longrightarrow$$
$$\longrightarrow H^0(C, \mathcal{O}_C(l)) \longrightarrow H^1(\mathbf{P}^3, I_C(l)) \longrightarrow 0,$$

we can see that $h^1(\mathbf{P}^3, I_C(1)) = 1$, $h^1(\mathbf{P}^3, I_C(2)) = 2$ and $h^0(\mathbf{P}^3, I_C(3)) \geq 1$. Suppose that $h^0(\mathbf{P}^3, I_C(3)) = 2$ and let S_1 and S_2 be cubic surfaces containing C. Then $S_1 \cap S_2 = C \cup D$, $\deg D = 2$ and $P_a(D) = -2$ by Lemma 1.3. Therefore D is a double line lying on a smooth cubic hypersurface.

Now look at the locus $\Sigma \subset G(1, 19)$ of pencils of cubic surfaces whose base locus consists of a curve of degree $d = 7$ with genus $g = 3$ and a double line, and at the map π_C, π_D of Σ to $I'_{7,3,3}$ and $H'_{2,-2,3}$ where the general members of $I'_{7,3,3}$ are smooth irreducible nondegenerate curves of degree $d = 7$ with genus $g = 3$ and the general members of $H'_{2,-2,3}$ are double lines with arithmetic genus $P_a = -2$. Then we know that $\dim I'_{7,3,3} = 28$ and $\dim H'_{2,-2,3} = 7$ (see [EH] p.34 and p.61). Let λ be the underlying line of D, then we have the following exact sequence

$$0 \longrightarrow \mathcal{O}_\lambda(1) \longrightarrow \mathcal{O}_D \longrightarrow \mathcal{O}_\lambda \longrightarrow 0.$$

Twisting by 3 and taking cohomology, we obtain

$$0 \longrightarrow H^0(\lambda, \mathcal{O}_\lambda(4)) \longrightarrow H^0(D, \mathcal{O}_D(3)) \longrightarrow$$
$$\longrightarrow H^0(\lambda, \mathcal{O}_\lambda(3)) \longrightarrow H^1(\lambda, \mathcal{O}_\lambda(4)) \longrightarrow .$$

Then we have $h^0(D, \mathcal{O}_D(3)) = h^0(\lambda, \mathcal{O}_\lambda(4)) + h^0(\lambda, \mathcal{O}_\lambda(3)) = 5 + 4 = 9$ since $h^1(\lambda, \mathcal{O}_\lambda(4)) = 0$. Therefore in the following exact sequence

$$0 \longrightarrow H^0(\mathbf{P}^3, I_D(3)) \longrightarrow H^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(3)) \longrightarrow$$
$$\longrightarrow H^0(D, \mathcal{O}_D(3)) \longrightarrow M(D)_3 \longrightarrow 0,$$

we have $h^0(\mathbf{P}^3, I_D(3)) = 11$ because $\dim M(D)_3 = 0$ (see [M2] p.179). Since the generic residual intersection of a pair of those cubics containing D is indeed a smooth curve of degree 7 with $g = 3$, π_D is surjective. And its fibers are open subsets of $G(1, 10)$ because $h^0(\mathbf{P}^3, I_D(3)) = 11$. Therefore $\dim \Sigma = 7 + \dim G(1, 10) = 7 + 18 = 25$. Similarly π_C is also surjective with fibers open in $G(1, 1)$ because $h^0(\mathbf{P}^3, I_C(3)) = 2$ and hence $\dim \Sigma = 28 + \dim G(1, 1) = 28$. This contradicts above result. Therefore $h^0(\mathbf{P}^3, I_C(3)) = 1$ i.e., $h^1(\mathbf{P}^3, I_C(3)) = 0$. Since $7 \leq 2 \cdot 3 + 3$, $M(C)_n = 0$ for $n \geq 4$ by Theorem 2.1.

THEOREM 2.5. *Let C and C' be a smooth irreducible nondegenerate curve of degree 7 with $g = 3$.*

(a) *C (resp. C') lies on a unique cubic surface S (resp. S').*

(b) *If S (resp. S') is smooth, then C (resp. C') has either a unique quinticsecant L_1 (resp. L'_1) or a unique line L_2 (resp. L'_2) which lies on S (resp. S') and disjoint from C (resp. C').*

(c) *If C and C' lie on a smooth cubic surface then $C \sim C'$ if and only if either $L_1 = L'_1$ or $L_2 = L'_2$.*

Proof. From Theorem 2.4, we know that $h^0(\mathbf{P}^3, I_C(3)) = 1$. If C lies on a smooth cubic surface S, then we see that C is linearly equivalent to one of the following types: $(a : b_i)$=(5:3,1,1,1,1,1), (7:3,3,3,3,1,1), (6:3,3,2,1,1,1), (7:4,3,2,2,2,1) and (8:4,4,3,2,2,2) by similar calculations as Theorem 2.3. We also see that the first two types have a unique quinticsecant and every 27 lines on S meets C. And the latter three types have no quinticsecant and a unique line on S which is disjoint from C by calculating intersection number of C with 27 lines on S. Hence (b) is proved.

To prove (c),we consider the following exact sequence

$$0 \longrightarrow H^0(\mathbf{P}^3, I_C(1)) \longrightarrow H^0(\mathbf{P}^3, I_C(2)) \longrightarrow$$
$$\longrightarrow H^0(\mathbf{P}^2, I_{C\cap H}(2)) \longrightarrow M(C)_1 \overset{\phi_1(L)}{\longrightarrow} M(C)_2 \longrightarrow$$

where H is the hyperplane defined by $L = 0$. Then $L^* \in V_{1,0}$ if and only if $h^0(\mathbf{P}^2, I_{C\cap H}(2)) = 1$ since $\dim M(C)_1 = 1$, $\dim M(C)_2 = 2$ and $h^0(\mathbf{P}^3, I_C(1)) = h^0(\mathbf{P}^3, I_C(2)) = 0$. This happens if and only if either five of the seven points of $C \cap H$ are collinear or seven points lie on an irreducible conic. Any plane H through the quinticsecant L_1 of C meets C in two more points and hence the seven points of $C \cap H$ lie on a reducible conic. Thus $L_1^* \in V_{1,0}$. Any plane H through the disjoint line L_2 from C meets S in the union of L_2 and a conic, so the seven points of $C \cap H$ must be coconical. Hence $L_2^* \in V_{1,0}$. Since there are a finite number of these, $\dim V_{1,0} = 1$. But this is the expected dimension of $V_{1,0}$ because $\dim M(C)_1 = 1$ and $\dim M(C)_2 = 2$. Therefore $\deg V_{1,0} = 1$ by Lemma 1.5. Since $V_{1,0}$ is the invariant of given liaison class, this completes the proof of (c).

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