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VANISHING THEOREM FOR COHOMOLOGY GROUPS AND STEIN-NESS PROBLEM

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1. Introduction

The Levi-problem is a very old problem in the theory of several complex variables and its original form was solved long ago. But some of the more general forms of the Levi problem still remain unsolved.

Because of Oka's characterization of domains of holomorphy by the plurisubharmonicity of $-\log d$, a domain Ω of $\mathbb{C}^n$ is Stein if and only if it is locally Stein in the sense that for every $x \in b\Omega$ there exists an open neighborhood U of x in $\mathbb{C}^n$ such that $U \cap \Omega$ is Stein. A natural question to raise is the relationship between Stein-ness and local Stein-ness for open subsets of a general complex space. For example, we have the following problem. In a Stein space, is a locally Stein open subset Stein ? This problem remains unsolved. The main difficulty is, of course, the lack of an analog of $-\log d$ for the case of a complex space. In the manifold case, Docquier-Grauert[9] proved that every locally Stein open subset of a Stein manifold is Stein.

Because of Oka's theorem, a domain in $\mathbb{C}^n$ (or a domain spread over $\mathbb{C}^n$) is Stein if it is the union of an increasing sequence of Stein open subsets. This was first proved by Behnke-Stein[4] prior to Oka's theorem. It is natural to ask when a manifold which is the union of an increasing sequence of Stein open subsets is Stein. By the result of Docquier-Grauert[9], such a manifold is Stein if it is an open subset of a Stein manifold. The answer to the following question is still unknown. Suppose $X_j \subset X_{j+1}$ are Stein open subsets of a Stein space X. Is $\bigcup_{j=1}^{\infty} X_j$ Stein ?

In the first part of this paper we prove that in the holomorphically separable complex space if the locus of holomorphic function is Stein, then the cohomology group $H^p(X, \mathcal{O})$ of finite dimension vanishes. In the last paragraph, by using the vanishing theorem, we get some partial

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answers to these Stein-ness questions. In this paper, X will denote a (paracompact) complex space of finite dimension.

2. The Vanishing Theorem

For every $f \in \mathcal{O}(X)$ let us set $Z(f) = \{x \in X : f(x) = 0\}$. Let x be a point of X and U_x be an open neighborhood of x in X. The point x is said to *be (holomorphically) defined by $\mathcal{O}(X)$ in U_x* if finitely many holomorphic functions $f_1, \ldots, f_s \in \mathcal{O}(X)$ exist such that

$$U_x \cap \bigcap_{j=1}^{s} Z(f_j) = \{x\}.$$

LEMMA 2.1([7]). *Let $(X, \mathcal{O})$ be a holomorphically separable complex space. Let D be a discrete subset of $(X, \mathcal{O})$. Let $\{X_j\}_{j \in I}$ be a countable family of irreducible closed analytic finite-dimensional subsets of X. For every $j \in I$ let us pick a point $x_j \in X_j$ and an open neighborhood U_j of x_j in X in which x_j is defined by $\mathcal{O}(X)$. For every $j \in I$ let us pick a point $y_j \in U_j \cap X_j$, $y_j \neq x_j$ (it is always possible). Then a holomorphic function $f \in \mathcal{O}(X)$ exists such that*

(1) *$f(x_j) \neq f(y_j)$ for every $j \in I$,*
(2) *$f(x) \neq 0$ for every $x \in D$,*
(3) *$\dim(Z(f) \cap X_j) < \dim X_j$ for every $j \in I$.*

We recall the notion of profondeur. Let R be a Noetherian ring and M an R-module of finite type. If $I \subset R$ is an ideal, a sequence of elements $f_1, \ldots, f_p \in I$ is called *a regular M-sequence* if f_k is not a zero-divisor of

$$M \Big/ \sum_{j=1}^{k-1} f_j M,$$

where $\sum_{j=1}^{0} f_j M = 0$. The maximum length of regular M-sequence is denoted by $prof_I\, M$(see [14]).

Definition 2.2. Let $(X, \mathcal{O})$ be a complex space and $\mathcal{F}$ be a coherent analytic sheaf on X. For $x \in X$ we define

$$prof_x \mathcal{F} = \begin{cases} \infty & \text{if } \mathcal{F}_x = 0 \\ prof_{\mathcal{O}_x} \mathcal{F}_x & \text{if } \mathcal{F}_x \neq 0 \end{cases}$$

and define *the singularity subvarieties of* $\mathcal{F}$ to be

$$S_k = \{x \in X : prof_x \mathcal{F} \leq k\}$$

for $k \geq 0$.

LEMMA 2.3([14]). *Suppose that* $(X, \mathcal{O})$ *is a complex space and let* $\mathcal{F}$ *be a coherent analytic sheaf on* X. *If* $f \in \mathcal{O}(X)$, f_x *is not a zero-divisor of* $\mathcal{F}$ *for every* $x \in X$ *if and only if*

$$\dim(Z(f) \cap S_{k+1}) \leq k$$

for all k.

THEOREM 2.4. *Let* $(X, \mathcal{O})$ *be a holomorphically separable complex space and* $\mathcal{F}$ *be a coherent analytic sheaf on* X. *Suppose that the following conditions are satisfied.*

(1) *For a suitable integer* $p \geq 1$, $H^p(X, \mathcal{F})$ *is finite-dimensional as a vector space on* $\mathbb{C}$.
(2) *If* f *is a holomorphic function on* X *which is non-constant on any finite dimensional irreducible component of* X, *then* $Z(f)$ *with its reducible structure is a Stein space.*

Then $H^p(X, \mathcal{F}) = 0$ *for the fixed* p.

Proof. We may assume that X is a connected space. Let $\{X_j\}_{j \in I}$ be the family of finite dimensional irreducible components of the singularity subvariety $S_k, k > 0$. Then I is countable. For every $j \in I$, let us choose points x_j, y_j as in Lemma 2.1 and let $f \in \mathcal{O}(X)$ be a holomorphic function chosen as in Lemma 2.1. For every holomorphic function $g \in \mathcal{O}(\mathbb{C})$ we shall denote by $g \circ f$ the holomorphic function on X which is denoted by composition of the functions f and g.

We first claim that for every choice of $g \in \mathcal{O}(\mathbb{C})$, $g \neq 0$, the germ $(g \circ f)_x$ is invertible in $\mathcal{O}_x$ for every $x \in X$. If g is not a constant function, g is open. Fix $k \geq 0$ and let T be an irreducible component of S_{k+1} of finite dimension. By Lemma 2.1(1) we know that f is nonconstant on T. Then f is open on T and $g \circ f$, as a mapping, is open on T. In particular, $g \circ f$ is nonconstant on T and we have

$$\dim(Z(g \circ f) \cap T) < \dim T \leq k + 1.$$

Hence $\dim(Z(g\circ f)\cap S_{k+1})\le k$ for all k. By Lemma 2.3, $(g\circ f)_x$ is not a zero-divisor of $\mathcal{F}_x$ for every $x\in X-S_0$.

Let us consider $g\in\mathcal{O}(\mathbb{C})$, $g\neq 0$, and the morphism $\pi:\mathcal{F}\longrightarrow(g\circ f)\mathcal{F}$ defined, for every $x\in X$, by $\pi_x(h_x)=(g\circ f)_x h_x$. Since $(g\circ f)_x$ is not a zero-divisor of $\mathcal{F}_x$ for every $x\in X-S_0$, π_x is injective for $x\notin S_0$. Thus the support of $ker\,\pi$ is discrete. Hence the homomorphism

$$\pi^*:H^p(X,\mathcal{F})\longrightarrow H^p(X,(g\circ f)\mathcal{F}),$$

induced by π, is an isomorphism.

Let $Z=\{x\in X:(g\circ f)(x)=0\}$ and let $\mathcal{O}_Z=(\mathcal{O}/(g\circ f)\mathcal{O})|_Z$. For $x\in X$, $g\circ f-(g\circ f)(x)$ has zero in X and hence we may assume that $Z\neq\emptyset$. Since $g\circ f$ is nonconstant on any irreducible component of X of finite dimension, by the condition (2), $(Z,\mathcal{O}_Z)$ is a Stein subspace(Grauert;[11]). Let us write $\mathcal{F}_Z=(\mathcal{F}/(g\circ f)\mathcal{F})|_Z$. Since $\mathcal{F}$ is $\mathcal{O}$-module, $\mathcal{F}_Z$ is $\mathcal{O}_Z$-module whence $H^p(X,\mathcal{F}_Z)=0$. Note that

$$H^p(X,\mathcal{F}/(g\circ f)\mathcal{F})\cong H^p(Z,\mathcal{F}_Z)$$

and hence we get $H^p(X,\mathcal{F}/(g\circ f)\mathcal{F})=0$. We consider the short exact sequence

$$0\to(g\circ f)\mathcal{F}\overset{i}{\hookrightarrow}\mathcal{F}\to\mathcal{F}/(g\circ f)\mathcal{F}\to 0$$

where i is the canonical injection. This short exact sequence induces the long exact sequence

$$\cdots\to H^p(X,(g\circ f)\mathcal{F})\to H^p(X,\mathcal{F})\to H^p(X,\mathcal{F}/(g\circ f)\mathcal{F})\to\cdots.$$

Since $H^p(X,\mathcal{F}/(g\circ f)\mathcal{F})=0$, the homomorphism

$$i^*:H^p(X,(g\circ f)\mathcal{F})\to H^p(X,\mathcal{F}),$$

induced by i, is surjective. Hence the endomorphism

$$\tau_g=i^*\circ\pi^*:H^p(X,\mathcal{F})\to H^p(X,\mathcal{F})$$

is surjective. Since $\dim H^p(X,\mathcal{F})<\infty,\tau_g$ is a $\mathbb{C}$-linear automorphism. Suppose that there is a cohomology class $\xi\in H^p(X,\mathcal{F}),\xi\neq 0$. Since $g\in\mathcal{O}(\mathbb{C}),g\neq 0$, is arbitrary, we can define the mapping

$$A:\mathcal{O}(\mathbb{C})\longrightarrow H^p(X,\mathcal{F})$$

by $A(0)=0$ and by $A(g)=\tau_g(\xi)$. This mapping is $\mathbb{C}$-linear. Notice that $A(g)=\tau_g(\xi)\neq 0$ for $g\neq 0$. Thus A is injective. Since $\mathcal{O}(\mathbb{C})$ is not finite dimensional over $\mathbb{C}$, this is a contradiction.

3. Stein-ness Problems

In this section $(X, \mathcal{O})$ will denote a holomorphically separable complex space.

LEMMA 3.1([5]). *Let Ω be an open subset of X. Suppose that X is Stein or $\Omega \Subset X$. Let f be a non-constant holomorphic function in any irreducible component of X and the following conditions are satisfied.*

(1) *Every holomorphic function on $\Omega \cap Z(f)$ can be extended on Ω.*
(2) *The subset $\Omega \cap Z(f)$ is Stein.*

Then Ω is a Stein space.

The following lemma was shown in [16]. We can simply derive the lemma by using the vanishing theorem.

LEMMA 3.2. *Let Ω be an open subset of X. Suppose that X is Stein or $\Omega \Subset X$. Suppose that either of the following conditions are satisfied.*

(1) *There exist open Stein subsets Ω_1, Ω_2 of X such that $\Omega = \Omega_1 \cup \Omega_2$.*
(2) *There exist an increasing sequence of Stein subsets $\Omega_1 \subset \Omega_2 \subset \dots$ such that $\Omega = \bigcup_{j=1}^{\infty} \Omega_j$.*

In each case, if $H^1(\Omega, \mathcal{O})$ is a finite-dimensional vector space over $\mathbb{C}$, then Ω is a Stein space.

Proof. (1) Let $\mathcal{F}$ be a coherent analytic sheaf on Ω. We consider the Mayer-Vietoris exact sequence

$$\cdots \to H^{p-1}(\Omega_1 \cap \Omega_2, \mathcal{F}) \to H^p(\Omega, \mathcal{F}) \to H^p(\Omega_1, \mathcal{F}) \oplus H^p(\Omega_2, \mathcal{F}) \to \cdots$$

for $p \geq 1$. Since Ω_1, Ω_2, and $\Omega_1 \cap \Omega_2$ are Stein, by Theorem B of Cartan,

$$H^p(\Omega_1, \mathcal{F}) = H^p(\Omega_2, \mathcal{F}) = H^p(\Omega_1 \cap \Omega_2, \mathcal{F}) = 0$$

for all $p \geq 1$. Hence we get $H^p(\Omega, \mathcal{F}) = 0$ for all $p \geq 2$.

If dim $X = 1$, since Ω is non-compact, the result is obvious. We shall prove by induction on dimension of X. Suppose that the result is true for the complex space of lower dimension of X. Let f be a non-constant holomorphic function on any finite-dimensional irreducible component of

X such that dim $Z(f) <$ dim X and $\Omega \cap Z(f) \neq \emptyset$. It is always possible by the Baire's category theorem. We consider the short exact sequence of coherent analytic sheaves on$X$

$$0 \to f\mathcal{O} \hookrightarrow \mathcal{O} \xrightarrow{\pi} \mathcal{O}/f\mathcal{O} \to 0$$

where π is the canonical projection. This induces the long exact sequence

$$\cdots \to H^1(\Omega, \mathcal{O}) \xrightarrow{\pi^*} H^1(\Omega, \mathcal{O}/f\mathcal{O}) \to H^2(\Omega, f\mathcal{O}) \to \cdots .$$

Since $f\mathcal{O}$ is a coherent analytic sheaf on Ω, $H^2(\Omega, f\mathcal{O}) = 0$. Hence the induced map

$$\pi^* : H^1(\Omega, \mathcal{O}) \to H^1(\Omega, \mathcal{O}/f\mathcal{O})$$

is surjective and $\dim H^1(\Omega, \mathcal{O}/f\mathcal{O}) < \infty$. Set $Z = Z(f)$. Since Z is closed,

$$H^1(\Omega, \mathcal{O}/f\mathcal{O}) \cong H^1(\Omega \cap Z, (\mathcal{O}/f\mathcal{O})|_Z)$$

so that $\dim H^1(\Omega \cap Z, \mathcal{O}_Z) < \infty$, where $\mathcal{O}_Z = (\mathcal{O}/f\mathcal{O})|_Z$. The subspaces $\Omega_1 \cap Z$ and $\Omega_2 \cap Z$ are open Stein subsets of $\Omega \cap Z$ and $\Omega \cap Z = (\Omega_1 \cap Z) \cup (\Omega_2 \cap Z)$. Since dim $Z <$ dim X, by the induction hypothesis, $(\Omega \cap Z, \mathcal{O}_Z)$ is a Stein space. By Theorem 2.4, $H^1(\Omega, \mathcal{O}) = 0$. From Lemma of Berg[5] every holomorphic function on $\Omega \cap Z(f)$ can be extended on Ω. By Lemma 3.1, Ω itself is a Stein space.

(2) Since Ω_j are Stein, by Theorem B of Cartan,

$$H^p(\Omega_j, \mathcal{O}) = H^{p+1}(\Omega_{j+1}, \mathcal{O}) = 0$$

for all $j \geq 1$ and $p \geq 1$. By the result of Andreotti-Vesentini[2], $H^p(\Omega, \mathcal{O}) = 0$ for all $p \geq 2$. By applying the same method as (1), we get the result.

PROPOSITION 3.3. *Let $\{\Omega_j\}_{j=1}^{\infty}$ be a sequence of open subsets of X such that $\Omega_j \cap \Omega_k = \emptyset$ for $|j-k| > 1$. Set $\Omega = \bigcup_{j=1}^{\infty} \Omega_j$. Suppose that X is Stein or that $\Omega \Subset X$. If $H^1(\Omega, \mathcal{O})$ is finite-dimensional, then Ω is a Stein space.*

Proof. Put $D_k = \bigcup_{j=1}^{k} \Omega_j$. Note that $D_{k+1} = D_k \cup \Omega_{k+1}$ and $D_k \cap \Omega_{k+1} = \Omega_k \cap \Omega_{k+1}$. We consider the Mayer-Vietoris exact sequence

$$\cdots \to H^1(D_{k+1}, \mathcal{O}) \to H^1(D_k, \mathcal{O}) \oplus H^1(\Omega_{k+1}, \mathcal{O}) \\ \to H^1(\Omega_k \cap \Omega_{k+1}, \mathcal{O}) \to \cdots$$

for $k = 2, 3, \ldots$. By Theorem B of Cartan, $H^1(\Omega_{k+1}, \mathcal{O}) = H^1(\Omega_k \cap \Omega_{k+1}, \mathcal{O}) = 0$ Hence $H^1(D_{k+1}, \mathcal{O}) \to H^1(D_k, \mathcal{O})$ is surjective. It follows that the restriction map $H^1(\Omega, \mathcal{O}) \to H^1(D_k, \mathcal{O})$ is surjective for any $k = 1, 2, \ldots$ [1, p.241]. So,

$$\dim H^1(D_k, \mathcal{O}) < \dim H^1(\Omega, \mathcal{O}) < \infty$$

for any $k = 1, 2, \ldots$.

Now we prove, inductively on k, that $D_k (k = 1, 2, \ldots)$ are Stein spaces. For $k = 1$ this is trivial because $D_1 = \Omega_1$ is Stein. Since $D_{k+1} = D_k \cup \Omega_{k+1}$ and $\dim H^1(D_{k+1}, \mathcal{O}) < \infty$, if D_k is Stein, then, by (1) of Lemma 3.2, D_{k+1} is Stein. By induction, $D_k (k = 1, 2, \ldots)$ are Stein.

Note that $\{D_k\}_{k=1}^{\infty}$ is an increasing sequence of Stein subsets such that $\Omega = \bigcup_{k=1}^{\infty} D_k$ and that $\dim H^1(\Omega, \mathcal{O}) < \infty$. By (2) of Lemma 3.2, Ω is a Stein space.

THEOREM 3.4([10]). *Let X be a (reduced) complex Stein space and let Ω be a relatively compact open subset of X. Suppose that the following conditions are satisfied.*

(1) *Ω is locally Stein.*
(2) *$H^1(\Omega, \mathcal{O}) = 0$.*
(3) *If f is holomorphic function on X which is non-constant on any finite dimensional irreducible component of X, then $\Omega \cap Z(f)$ with its reduced structure is a Stein space.*

Then Ω is Stein.

In [8] Coltoiu and Mihalache dropped the condition (3). Applying the result of Lemma 3.2 we sharpen Theorem 3.4.

THEOREM 3.5. *Let X be a (reduced) Stein space of finite dimension and let Ω be an open subset of X. Suppose that the following conditions are satisfied.*

(1) *Ω is locally Stein.*
(2) *$H^1(\Omega, \mathcal{O})$ is a finite-dimensional vector space over $\mathbb{C}$.*

Then Ω is a Stein space.

Proof. Since Ω is locally Stein, there is a Stein covering $\mathcal{U} = \{U_\alpha\}$ of X such that $U_\alpha \cap \Omega$ is Stein for any α.

From the result of Stehlè[15, p.167] we get a Stein covering $\mathcal{B} = \{B_j\}_{j=1}^{\infty}$ of X which is a refinement of $\mathcal{U}$ and such that $C_j = \bigcup_{i\geq j} B_i$ is Stein for any $j = 1, 2, \ldots$. Let $\Omega_j = C_j \cap \Omega$ and $D_j = B_j \cap \Omega$.

We first claim that for any j the restriction map

$$H^1(\Omega, \mathcal{O}) \to H^1(\Omega_j, \mathcal{O})$$

is surjective. We consider the Mayer-Vietoris exact sequence

$$\cdots \to H^1(\Omega_{j+1}, \mathcal{O}) \to H^1(\Omega_j, \mathcal{O}) \oplus H^1(D_{j+1}, \mathcal{O}) \\ \to H^1(\Omega_j \cap D_{j+1}, \mathcal{O}) \to \cdots .$$

Note that D_{j+1} and $\Omega_j \cap D_{j+1}$ are Stein. Therefore $H^1(D_{j+1}, \mathcal{O}) \to H^1(\Omega_j, \mathcal{O})$ is surjective. It follows that the restriction map $H^1(\Omega, \mathcal{O}) \to H^1(\Omega_j, \mathcal{O})$ is surjective for any $j = 1, 2, \ldots$ [1, p.241]. Hence

$$\dim H^1(\Omega_j, \mathcal{O}) < \dim H^1(\Omega, \mathcal{O}) < \infty$$

for any $j = 1, 2, \ldots$.

Now we shall prove, inductively on j, that Ω_j are Stein. For $j = 1$, $\Omega_1 = B_1 \cap \Omega$ is Stein. Notice that $\Omega_{j+1} = \Omega_j \cup D_{j+1}$ and $\dim H^1(\Omega_{j+1}, \mathcal{O}) < \infty$. If Ω_j is Stein, by Lemma 3.2, Ω_{j+1} is Stein. By induction, $\Omega_j (j = 1, 2, \ldots)$ are all Stein. Note that $\Omega_1 \subset \Omega_2 \subset \cdots$ is an increasing sequence of open Stein subsets of X. Since $\Omega = \bigcup_{j=1}^{\infty} \Omega_j$ and $\dim H^1(\Omega, \mathcal{O}) < \infty$, by (2) of Lemma 3.2, Ω is Stein.

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