Comm. Korean Math. Soc. **8** (1993), No. 4, pp. 669–678

NUMERICAL RANGES IN NON UNITAL NORMED ALGEBRAS

YOUNGOH YANG

1. Introduction

Let A denote a unital normed algebra over a field $K = \mathbb{R}$ or $\mathbb{C}$ and let e be the identity of A. Given $a \in A$ and $x \in A$ with $\|x\| = 1$, let

$$V(A, a, x) = \{f(ax) : f \in A', f(x) = 1 = \|f\|\}.$$

Then the (Bonsall and Duncan) numerical range of an element $a \in A$ is defined by

$$V(a) = \cup\{V(A, a, x) : x \in A, \|x\| = 1\},$$

where A' denotes the dual of A. In [2], $V(a) = \{f(a) : f \in A', f(e) = 1 = \|f\|\}$. (see [2], [3] for details.)

We have two limitations in this numerical range: First this definition of $V(A, a)$ is dependent on the identity. There are many normed algebras which do not possess an identity. Therefore it is of some interest to make the notion of relative numerical range identity-free.

The second limitation is in the definition itself. For $a \in A$, a normed algebra, the scalars comprising the numerical range of a are of the form $f(ax)$ where $x \in A, f \in A'$, and $1 = \|x\| = \|f\| = f(x)$. No consideration is given to scalars of the form $f(xa)$, and as will be seen, these are significant if progress is to be made.

In this paper we introduce the notion of right(left) relative numerical range $V_x^R(A, a)(V_x^L(A, a))$ of an element a of a non unital normed algebra A relative to $x \in A$. (See Definition 2.1) If $x = e$, the identity of A and $\|e\| = 1$, then $V_x^R(A, a)$ coincides with $V(a)$. Thus this concept extends the (Bonsall and Duncan) numerical range. Among the results, it is

Received April 29, 1993.

This paper was supported (in part) by NON DIRECTED RESEARCH FUND, Korea Research Foundation, 1992.

shown that our numerical range is a compact convex subset of K. Also we give a sufficient condition for our numerical range to be a singleton set $\{1\}$.

Further, we show that the relative numerical range of an element in a normed algebra is invariant under certain algebra homomorphism. An example is given to show that the invariance of the relative numerical range under homomorphism ϕ does not imply that ϕ is an isometry. Also we introduce the concept of regular norm on a normed algebra and give a sufficient condition for a normed algebra to have regular norm in terms of our relative numerical range.

Throughout this paper let A be a non unital normed algebra over a field K ($\mathbb{R}$ or $\mathbb{C}$).

2. Relative numerical ranges of elements

DEFINITION 2.1. Let A be a normed algebra over the field $K = \mathbb{R}$ or $\mathbb{C}$, and A' its dual. For $x \in A$, we write

$$D(A,x) = \{f \in A' : \|f\| = 1, f(x) = \|x\|\}.$$

The right relative numerical range of $a \in A$ relative to x is defined to be $V_x^R(A,a) = \{f(ax) : f \in D(A,x)\}$. The left relative numerical range of $a \in A$ relative to x is defined to be $V_x^L(A,a) = \{f(xa) : f \in D(A,x)\}$. The relative numerical range of a relative to x is defined to be $V_x(A,a) = V_x^R(A,a) \cup V_x^L(A,a)$. The right relative numerical radius of a relative to x is defined by $v_x^R(a) = \sup\{|\lambda| : \lambda \in V_x^R(A,a)\}$. The left relative numerical radius of a relative to x is defined by $v_x^L(a) = \sup\{|\lambda| : \lambda \in V_x^L(A,a)\}$. The relative numerical radius of a relative to x is defined by $v_x(a) = \max\{v_x^R(a), v_x^L(a)\}$.

Note that the set $D(A,x)$ is nonempty by the Hahn-Banach Theorem, and so $V_x^R(A,a)$ and $V_x^L(A,a)$ are nonempty. If A is commutative, then $V_x^R(A,a) = V_x^L(A,a) = V_x(A,a)$ as $f(ax) = f(xa)$. If $b = e$(identity of A) with $\|e\| = 1$, then $V_e(A,a) = V(a)$, where $V(a)$ denotes the (Bonsall and Duncan) numerical range of a [2]. Thus the concept of numerical range is a special case of that of relative numerical range.

LEMMA 2.2. *Let $a, b, x \in A$ and $\alpha, \beta \in K$. Then*

(1) $V_x(A, \alpha a + \beta b) \subseteq \alpha V_x(A, a) + \beta V_x(A, b)$, *and* $V_x(A, \alpha a) = \alpha V_x(A, a)$.

(2) $v_x(a + b) \leq v_x(a) + v_x(b)$ *and* $v_x(\alpha a) = |\alpha| v_x(a)$.

(3) $v_x(a) \leq \max\{\|ax\|, \|xa\|\}$.

Proof. (1) Let $f \in D(A, x)$. Since $f((\alpha a + \beta b)x) = \alpha f(ax) + \beta f(bx)$ and $f((\alpha a)x) = f(\alpha(ax)) = \alpha f(ax)$, $V_x^R(A, \alpha a + \beta b) \subseteq \alpha V_x^R(A, a) + \beta_x^R(A, a)$ and $V_x^R(A, \alpha a) = \alpha V_x^R(A, a)$

Similar statements hold in terms of V^L, hence taking unions

$$V_x(A, \alpha a + \beta b) \subseteq \alpha V_x(A, a) + \beta V_x(A, b) \text{ and } V_x(A, \alpha a) = \alpha V_x(A, a).$$

(2) This follows from (1).

(3) $\lambda \in V_x(A, a)$ implies $\lambda = f(ax)$ or $f(xa)$ for some $f \in D(A, x)$. Hence $|\lambda| = |f(ax)| \leq \|f\|\|ax\| = \|ax\|$ or $|\lambda| = |f(xa)| \leq \|xa\|$.

We note that the inclusion relation in (1) cannot be replaced by the equality in general e.g. take $a = -b$.

LEMMA 2.3. *Let $a, x \in A$. Then*

(1) *$D(A, x)$ is a weak* compact convex subset of A'.*

(2) *$V_x^R(A, a)$ and $V_x^L(A, a)$ are compact convex subsets of K, hence $V_x(A, a)$ is a compact subset of K.*

Proof. (1) Let $f, g \in D(A, x)$ and let λ be any number in [0,1]. Then $\|\lambda f + (1 - \lambda)g\| \leq \lambda\|f\| + (1 - \lambda)\|g\| = 1$ and $(\lambda f + (1 - \lambda)g)(x) = \|x\|$. So $\|\lambda f + (1 - \lambda)g\| = 1$ and $\lambda f + (1 - \lambda)g \in D(A, x)$. Therefore $D(A, x)$ is convex.

Define $e_x(f) = f(x)$. Then e_x is *weak** continuous, i.e., continuous in the pointwise convergence topology on A'. By [3], $D \equiv \{f \in A' : \|f\| \leq 1\}$ is *weak** compact. Hence

$$D(A, x) = D \cap e_x^{-1}(\{\|x\|\})$$

is a *weak** closed subset of D and so is *weak** compact.

(2) Define $e_{ax}(f) = f(ax)$. e_{ax} is *weak** continuous, so $V_x^R(A, a) = e_{ax}(D(A, x))$ is a compact subset of K. As e_{ax} is linear and $D(A, x)$ is convex, $V_x^R(A, a)$ is convex. Similarly $V_x^L(A, a)$ is a compact convex subset of K. Hence $V_x(A, a) = V_x^R(A, a) \cup V_x^L(A, a)$ is a compact subset of K.

THEOREM 2.4. *If B is a subalgebra of a normed algebra A and $b, x \in B$, then $V_x(B,b) = V_x(A,b)$.*

Proof. The mapping $f \longrightarrow f|_B$ takes $D(A,x)$ into $D(B,x)$ because $\|f\| = 1, f(x) = \|x\|$ implies $f|_B(x) = \|x\|, \|f|_B\| = 1$. Recall that $\|f|_B\| \leq \|f\|$. Given $g \in D(B,x)$, the Hahn- Banach Theorem implies there exists an $f \in A'$ such that $f|_B = g$ and $\|f\| = \|g\|$. So $f \in D(A,x)$. Hence $V_x^R(B,b) = V_x^R(A,b)$ and $V_x^L(B,b) = V_x^L(A,b)$. Taking unions $V_x(B,b) = V_x(A,b)$.

THEOREM 2.5. *Let $a \in A$ and let x be any nonzero element of A. Then*

(1) *If $ax = x$, then $V_x^R(A,a) = \{\|x\|\}$.*
(2) *If $V_x^R(A,a) = \{\|x\|\}$, then either $ax = x$ or $0 < dist(x, Kax) < \|x\|$.*

Proof. (1) This follows from the definition.

(2) Suppose that $V_x^R(A,a) = \{\|x\|\}$. First we note that $dist\,(x, Kax) = \inf \|x - \lambda ax\| \leq \|x\|$. If $ax \neq x$ and $x \in Kax$, then $x = \lambda ax, \lambda \neq 1 (\lambda \in K)$ implies that $f(x) = \lambda f(ax) = \lambda \|x\|$ for any $f \in D(A,x)$. This is a contradiction because $f(x) \neq 1$. Hence $ax = x$. If $x \notin Kax$ and $dist\,(x, Kax) = \|x\|$, then by ([3], p. 82 or [4], p.64) there exists $f \in A'$ such that $\|f\| = 1, f(x) = \|x\|$ and $f(ax) = 0$. This is a contradiction to our hypothesis. Hence $0 < dist(x, Kax) < \|x\|$.

We have the similar statement for the left relative numerical range:

THEOREM 2.6. *Let $a \in A$ and let x be any nonzero element of A. Then*

(1) *If $xa = x$, then $V_x^L(A,a) = \{\|x\|\}$.*
(2) *If $V_x^L(A,a) = \{\|x\|\}$, then either $xa = x$ or $0 < dist(x, Kxa) < \|x\|$.*

COROLLARY 2.7. *If $a \in A$, and $a^2 = a$, then $V_a^R(A,a) = V_a^L(A,a) = V_a(A,a) = \{\|a\|\}$.*

LEMMA 2.8. *Let $a, b, x \in A$, and let $N_\epsilon = N(0,\epsilon)$. If $\|a - b\| < \epsilon$, then $V_x^R(A,b) \subseteq V_x^R(A,a) + \|x\|N_\epsilon$ and $V_x^R(A,a) \subseteq V_x^R(A,b) + \|x\|N_\epsilon$.*

Proof. Let $\lambda \in V_x^R(A,b)$. There exists $f \in D(A,x)$ such that $\lambda = f(bx)$. Thus

$$\begin{aligned} |\lambda - f(ax)| = |f(bx) - f(ax)| &= |f((b-a)x)| \leq \|f\|\|b-a\|\|x\| \\ &= \|b-a\|\|x\| < \|x\|\epsilon. \end{aligned}$$

So $\lambda \in V_x^R(A,a) + \|x\|N_\epsilon$. Similarly $V_x^R(A,a) \subseteq V_x^R(A,b) + \|x\|N_\epsilon$.

REMARK 2.9. The previous lemma is true for the left relative numerical range by a similar proof.

THEOREM 2.10. *Let $a, b, x \in A$, and let $N_\epsilon = N(0,\epsilon)$. If $\|a-b\| < \epsilon$, then $V_x(A,b) \subseteq V_x(A,a) + \|x\|N_\epsilon$ and $V_x(A,a) \subseteq V_x(A,b) + \|x\|N_\epsilon$.*

Proof. By the Lemma 2.8 and Remark 2.9, $V_x^R(A,a) \subseteq V_x^R(A,b) + \|x\|N_\epsilon$ and $V_x^L(A,a) \subseteq V_x^L(A,b) + \|x\|N_\epsilon$. Hence

$$\begin{aligned} V_x(A,a) &= V_x^R(A,a) \cup V_x^L(A,a) \\ &\subseteq (V_x^R(A,b) + \|x\|N_\epsilon) \cup (V_x^L(A,b) + \|x\|N_\epsilon) \\ &= \{V_x^L(A,b) \cup V_x^R(A,b)\} + \|x\|N_\epsilon \\ &= V_x(A,b) + \|x\|N_\epsilon. \end{aligned}$$

Therefore

$$V_x(A,a) \subseteq V_x(A,b) + \|x\|N_\epsilon.$$

Exchanging a and b,

$$V_x(A,b) \subseteq V_x(A,a) + \|x\|N_\epsilon.$$

Consider a pair of compact subsets of the complex plane, M and N and define $d(M,N) = \inf\{\epsilon : M \subseteq N + N_\epsilon,\ N \subseteq M + N_\epsilon\}$. Then for $a, b, x \in A$ we can consider $d(V_x^R(A,a), V_x^R(A,b))$ as a metric, the "Hausdorff metric" on sets associated with a and b.

THEOREM 2.11. *For each $x \in A$, $V_x^R(\cdot)$ is a continuous from A endowed with the norm topology to the set of compact subsets of $\mathbb{C}$, endowed with the Hausdorff metric topology. Also $v_x^R(\cdot)$ is a continuous real-valued function on A.*

Proof. Let $a, b \in A$ with $\|a-b\| < \epsilon$. Then by Theorem 2.8,

$$d(V_x^R(A,a), V_x^R(A,b)) \leq \epsilon,$$

and $V_x^R(\cdot)$ is continuous.

Also $v_x^R(a) \leq v_x^R(b)+\epsilon$ and $v_x^R(b) \leq v_x^R(a) \leq \epsilon$ imply $|v_x^R(a)-v_x^R(b)| \leq \epsilon$. So $v_x^R(\cdot)$ is a continuous function.

This theorem is true for the left relative numerical range V_x^L and numerical radius v_x^L.

The following theorem gives the invariance of relative numerical ranges under isometric algebraic homomorphism.

THEOREM 2.12. *Let ϕ be an isometric algebraic homomorphism of a normed algebra A into a normed algebra B. Then*

$$V_{\phi(x)}^R(B, \phi(a)) = V_x^R(A, a)$$

for all $a \in A$.

Proof. Let $\lambda \in V_{\phi(x)}^R(B, \phi(a))$. Then there exists $g \in D(B, \phi(x))$ such that $\lambda = g(\phi(ax))$. Define f on A by $f(z) = g(\phi(z)), z \in A$. Clearly, f is linear and $\|f\| \leq 1$. Since ϕ is an isometry, $\|\phi(x)\| = \|x\|$ implies $\|f\| = 1$ and so $\lambda = f(ax) \in V_x^R(A, a)$.

Conversely if $\mu \in V_x^R(A, a)$, then there exists $f \in D(A, x)$ such that $\mu = f(ax)$. Define g on $\phi(A) = \{\phi(z) : z \in A\}$ by $g(\phi(z)) = f(z), z \in A$. Then again we see that g is a bounded linear functional on $\phi(A)$ with $\|g\| = 1$ because ϕ is an isometry. By ([3], p.81 or [4], p. 63) g can be extended to a bounded linear functional h on B with $\|h\| = \|g\| = 1$ and $h(\phi(x)) = f(x)$. Hence $\mu = f(ax) = g(\phi(ax)) = h(\phi(ax)) \in V_{\phi(x)}^R(B, \phi(a))$.

REMARK 2.13. The previous theorem is true for the left relative numerical range.

COROLLARY 2.14. *Let ϕ be an isometric algebraic homomorphism of a normed algebra A into a normed algebra B. Then*

$$V_{\phi(x)}(B, \phi(a)) = V_x(A, a)$$

for all $a \in A$.

We note that the invariance of relative numerical ranges under an algebraic homomorphism in Theorem 2.12 does not imply isometry. For we consider an algebra A having divisors of zero, $a \neq 0, b \neq 0, ab = 0$ and a zero homomorphism ϕ of A with an arbitrary algebra B. Then $V_b^R(A, a) = \{0\} = V_{\phi(b)}^R(B, \phi(a))$ but ϕ is not an isometry.

3. Topological divisors of zero and regular norm

DEFINITION 3.1. Let A be a normed algebra. An element $a \in A$ is called a left(right) topological divisor of zero provided there exists a sequence $\{z_n\}$ in A such that $\|z_n\| = 1$ for all n and $zz_n \longrightarrow 0(z_n z \longrightarrow 0)$. An element which is either a left or right topological divisor of zero is called a topological divisor of zero. We denote the set of all left(right) divisors of zero in A, and the set of all left(right) topological divisors of zero in A by $ldoz(A)(rdoz(A))$, and $ltdoz(A)(rtdoz(A))$ respectively.

PROPOSITION 3.2. *Let A be a normed algebra over K. Then*

(1) *$a \notin ldoz(A)$ implies $\overline{Aa} = Aa$ if and only if $a \notin ltdoz(A)$.*
(2) *$a \notin rdoz(A)$ implies $\overline{aA} = aA$ if and only if $a \notin rtdoz(A)$.*

Proof. Let $L_a(x) = ax, R_a(x) = xa$. Then L_a and R_a are one to one and have continuous inverse if and only if a is not a left or right topological divisor of zero, ([7], p.21).

THEOREM 3.3. *Let a be any element of a normed algebra A. Then*

(1) *$a \in ldoz(A)$ implies $0 \in V_x^R(Aa)$ for some $x \in A$.*
(2) *$a \in rdoz(A)$ implies $0 \in V_x^L(Aa)$ for some $x \in A$.*

Proof. (1) By hypothesis, there is a nonzero element $x \in A$ such that $ax = 0$. By the Hahn Banach theorem, there is $f \in A'$ such that $f(x) = \|x\|$, and $\|f\| = 1$. So $0 = f(0) = f(ax) \in V_x^R(A, a)$.
(2) By symmetry this follows from (1).

DEFINITION 3.4. Let A be a normed algebra over K. An element $a \in A$ is said to have right(left) regular norm if

$$\|a\| = \sup_{\|x\| \leq 1} \|ax\| \quad (\|a\| = \sup_{\|x\| \leq 1} \|xa\|).$$

If each $a \in A$ has right(left) regular norm, then A is said to have right(left) regular norm.

PROPOSITION 3.5. *Let A be a normed algebra over K. If A has a right(left) approximate identity $\{u_\alpha\}_{\alpha \in I}$ with $\sup_\alpha \|u_\alpha\| \le 1$, then A has right(left) regular norm.*

Proof. Suppose that A has a right approximate identity. Then $|\|a\| - \|au_\alpha\|| \le \|a - au_\alpha\|$ for each $a \in A$, and so $\|au_\alpha\| \longrightarrow \|a\|$ as $n \to \infty$. Hence $\|a\| \le \sup_\alpha \|au_\alpha\| \le \sup_{\|x\| \le 1} \|ax\| \le \|a\|$. Thus A has right regular norm. By the similar proof A has left regular norm if A ha a left approximate identity.

In particular, if A has identity $e, \|e\| = 1$, then A has right and left regular norm.

The following example shows that a normed algebra with a regular norm need not have a bounded approximate identity.

EXAMPLE 3.6. Let Δ be the closed unit disc in $\mathbb{C}$ and let $A(\Delta)$ denote the disc algebra. Then $A(\Delta)$ is a commutative normed algebra with identity, and so has (right and left) regular norm. Hence $A_0 = \{f \in A(\Delta) : f(0) = 0\}$ is a commutative normed algebra with identity and so has (right and left) regular norm.

Let $g \in A_0$ such that $g(\lambda) = \lambda$. The functional F on A_0 defined by $F(f) = f'(0)$ (derivative of f evaluated at 0) is clearly continuous. If $\{u_\alpha\}$ is an approximate identity of A_0, then $\lim_\alpha F(u_\alpha g) = F(g) = 1$. But this gives a contradiction since $F(u_\alpha g) = u_\alpha(0) + g(0)F(u_\alpha) = 0$ for all α. Thus A_0 has no approximate identity (bounded or unbounded).

We give a sufficient condition in terms of relative numerical ranges for a normed algebra to have a regular norm.

THEOREM 3.7. *Let A be a normed algebra. If there is $x \in A$ such that $\|x\| = 1$ and $V_x^R(A, a) = \{\|a\|\}(V_x^L(A, a) = \{\|a\|\})$, then a has a right(left) regular norm.*

Proof. There is an $f \in D(A, x)$ such that $f(ax) = \|a\|$, and so

$$\|a\| = |f(ax)| \le \|ax\| \le \|a\|\|x\| = \|a\|.$$

Thus a has a right regular norm. A similar statement holds in terms of V^L.

COROLLARY 3.8. *Let A be a normed algebra. If there is $x \in A$ such that $\|x\| = 1$ and $V_x(A, a) = \{\|a\|\}$, then a has a right and left regular norm.*

THEOREM 3.9. *Let A be a normed algebra with right regular norm. Suppose $B(A)$ is the algebra of all bounded linear operators on A. Then for $a, x \in A$,*

$$V_x^R(A, a) = V_{T_x}^R(B(A), T_a),$$

where T_z is the left regular representation on A.

Proof. Suppose A^+ is the unitization of A when A has a right regular norm. Let $(a, \lambda) \in A^+$ and define an operator $T_{(a,\lambda)}$ on A by

$$T_{(a,\lambda)}(z) = az + \lambda z, (z \in A, \lambda \in \mathbb{C})$$

Then this operator is clearly linear and bounded. Also the function $\phi : A^+ \longrightarrow B(A)$ defined by $\phi(a, \lambda) = T_{(a,\lambda)}$ is an algebra homomorphism. In fact this homomorphism is a monomorphism.

Define the norm $\|\cdot\|_+$ on A^+ by $\|(a, \lambda)\|_+ = \|T_{(a,\lambda)}\|$. Then since A has right regular norm, we have

$$\begin{aligned} \|(a, 0)\|_+ = \|T_{(a,0)}\| &= \sup_{\|x\| \le 1} \|T_{(a,0)}(x)\| \\ &= \sup_{\|x\| \le 1} \|ax\| = \|a\|. \end{aligned}$$

This proves the extension of the original norm on A to the norm on A^+ and so we have an isometric algebra homomorphism from A to $B(A)$. By Theorem 2.11,

$$V_x^R(A, a) = V_{\phi(x)}^R(B(A), \phi(a)) = V_{T_x}^R(B(A), T_a).$$

REMARK 3.10. Let A be a normed algebra with left regular norm. Suppose $B(A)$ is the algebra of all bounded linear operators on A. Then for $a, x \in A$,

$$V_x^L(A, a) = V_{T_x}^L(B(A), T_a),$$

where T_x is the right regular representation on A.

References

1. F. F. Bonsall and J. Duncan, *Numerical ranges of operators on normed spaces and of elements of normed algebras,* Cambridge University Press, Cambridge, 1971.
2. ———, *Numerical ranges II,* Cambridge University Press, Cambridge, 1971.
3. J. B. Conway, *A course in functional analysis,* Springer-Verlag, New York, 1985.
4. N. Dunford and J. Schwartz, *Linear operators Part I,* Interscience, New York, 1958.
5. R. Larsen, *Banach algebras; an introduction,* Marcel Dekker, New York, 1973.
6. Martinez, *Numerical range without Banach algebras,* Illinois J. Math. **29** (1985), 609–625.
7. C. E. Rickart, *General theory of Banach algebras,* Van Nostrand, New York, 1960.

Department of Mathematics
Cheju National University
Cheju 690-756, KOREA