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SPECTRAL TYPES OF SKEWED IRRATIONAL ROTATIONS

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0. Introduction

Throughout this article the unit circle $\mathbb{T} = \{e^{2\pi is} : s \ \text{real}\}$ is identified with the half-open interval $[0,1)$. Given an irrational number $\theta \in \mathbb{T}$ and an interval $I \subset \mathbb{T}$, we define for $n \geq 1$ an integer-valued function $S_{n;I}(x) \equiv \sum_{j=0}^{n-1} \chi_I(x + j\theta)$ where χ_I is the characteristic function of I. Then the Kronecker-Weyl theorem states that $\lim_{n\to\infty} \frac{1}{n} S_{n;I}(x) = m(I)$ for every x where m is the normalized Lebesgue measure on $\mathbb{T}$. If $y_n \equiv S_{n;I}(0) \pmod 2$, $y_n \in \{0,1\}$, then one might expect that the limit $\mu_\theta(I) \equiv \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} y_n$ exists and is equal to $\frac{1}{2}$. But W.A. Veech [16] proved that the existence of $\mu_\theta(I)$ depends only on the length of the interval I for fixed θ and that $\mu_\theta(I)$ exists for every interval $I \subset \mathbb{T}$ if and only if θ has bounded partial quotients in its continued fraction expansion, which is contrary to our intuition. Recall that a function f is called a *coboundary* if it is of the form $f(x) = \overline{g(x)} g(x+\theta)$ for some measurable function g of modulus 1 almost everywhere. In the case when θ has bounded partial quotients the limit $\mu_\theta(I)$ exits and is equal to $\frac{1}{2}$ if $\exp(\pi i \chi_I)$ is not a constant multiple of a coboundary.

In this article, we investigate the behavior of the sequence $\int_0^1 \exp(\pi i S_{n;I}(x))\, dx$. Since $\exp(\pi i S_{n;I}(0)) = 1 - 2y_n$, we see that $\mu_\theta(I) = \frac{1}{2}$ is equivalent to the condition

$$\lim_{N\to\infty} \frac{1}{N} \sum_{1}^{N} \exp(\pi i S_{n;I}(0)) = 0.$$

Note that $\exp(\pi i S_{n;I}(x)) = \exp(\pi i S_{n;I-x}(0))$ and that the existence of the limit $\mu_\theta(I)$ depends only on the length of the interval I. Hence

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the limit of the sequence $\frac{1}{N}\sum_1^N \exp(\pi i S_{n;I}(x))$ is 0 for any $x \in \mathbb{T}$ if and only if the same is true for $x = 0$. Thus if $\mu_\theta(I)$ equals $\frac{1}{2}$, then $\lim \frac{1}{N}\sum_1^N \int_0^1 \exp(\pi i S_{n;I}(x))\,dx = 0$. In Section 1 we find the necessary and sufficient condition for which

$$\lim_{N\to\infty} \frac{1}{N}\sum_1^N \left| \int_0^1 \exp(\pi i S_{n;I}(x))\,dx \right| = 0.$$

The advantage of considering $\int_0^1 \exp(\pi i S_{n;I}(x))\,dx$ instead of y_n is that we can use spectral theory of unitary operators. And we can estimate the extent of irregularity of convergence to zero. See Theorem 1, (iii).

Define a unitary operator $U : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ by $(Uf)(x) = A(x)f(x+\theta)$ where $|A(x)| \equiv 1$ almost everywhere. Then for a constant function 1, we have $(U^n1)(x) = A(x)A(x+\theta)\cdots A(x+(n-1)\theta))$ for $n \geq 1$. Let P be the spectral measure on $\mathbb{T}$ such that $U = \int_0^1 e^{2\pi i x}\,dP(x)$. The spectral properties of P depend on $A(x)$. For example,

$$\int_0^1 A(x)A(x+\theta)\cdots A(x+(n-1)\theta))\,dx = (U^n1,1) = \int_0^1 e^{2\pi i n x}\,d(P(x)1,1)$$

is the n-th Fourier-Stieltjes coefficient of the positive Borel measure $\mu_1 : E \mapsto (P(E)1,1)$ where E is a measurable set in $\mathbb{T}$. It is known that the spectral type of P is pure, i.e., it is either purely discrete or purely singular continuous or purely absolutely continuous. In other words, the maximal spectral type of U is pure. For, if we let

$$L^2(\mathbb{T}) = H_{ac} \oplus H_{sc} \oplus H_d$$

be the decomposition into three U-invariant subspaces such that U has absolutely continuous spectrum in H_{ac} and singular continuous spectrum in H_{sc} and discrete spectrum in H_d, then each subspace is invariant under the multiplication by $e^{2\pi i x}$ and the irrational rotation by θ. Hence each subspace is either the whole space $L^2(\mathbb{T})$ or the trivial subspace $\{0\}$. It can easily be shown that $A(x)$ is a constant multiple of a coboundary if and only if P is discrete. It is also known that if P is absolutely continuous, then it is also Lebesgue, in other words, there exists $f \in$

$L^2(\mathbb{T})$ such that the positive Borel measure $\mu_f : E \mapsto (P(E)f, f)$ is the Lebesgue measure on $\mathbb{T}$. For the details, see Helson ([7],[8]).

In Section 1, an integral version of Veech's theorem is obtained by classifying the spectral types of U where $\phi(x)$ is a step function. In Section 2, regarding $A(x) = e^{2\pi i\phi(x)}$ as a map from the unit circle into itself, we show that the winding number of $A : \mathbb{T} \to \mathbb{T}$ classifies the spectral types of U for sufficiently smooth functions A.

We use the continued fractions as a computational tool. Only the following elementary facts will be used : Let $\theta = [a_1, a_2, ..., a_k, ...\ ]$ be the continued fraction expansion of an irrational number $\theta \in \mathbb{T}$, where $a_1, a_2, ...$ are positive integers which are called partial quotients, and put $m_k/n_k = [a_1, a_2, ..., a_k]$ where m_k and n_k are relatively prime. Then we have $|\theta - (m_k/n_k)| < 1/(\sqrt{5}n_k^2)$ for infinitely many pairs of (m_k, n_k). For every irrational number θ with bounded partial quotients, there exists a positive constant c for which the inequality $|\theta - (m/n)| < c \cdot 1/n^2$ has no solution in integers m and n, $n > 0$. On the other hand, for an irrational number θ with an unbounded sequence of partial quotients the inequality has an infinite set of solutions for any arbitrary $c > 0$. The irrational numbers with bounded partial quotients form a set of measure zero. For a standard reference, see [10].

1. Spectrum and uniform distribution

An operator U is defined as in Section 0. Let us consider the case when $A(x)$ is given by $A(x) = \exp(\pi i \chi_I(x))$ where χ_I is the characteristic function of an interval $I \subset \mathbb{T}$. In this case, the unitary operator U on $L^2(\mathbb{T})$ is given by $(Uf)(x) = \exp(\pi i \chi_I(x)) f(x+\theta)$. So for some spectral measure P defined on $\mathbb{T}$, $U^n = \int_0^1 e^{2\pi i n x} dP(x)$. Since for $n \geq 1$

$$(U^n 1)(x) = \exp(\pi i \sum_{k=0}^{n-1} \chi_I(x + k\theta)) = \exp(\pi i S_{n;I}(x)),$$

we see that

$$\begin{aligned}\int_0^1 \exp(\pi i S_{n;I}(x))\,dx &= \int_0^1 (U^n 1)(x)\,dx \\ &= (U^n 1, 1) \\ &= \int_0^1 e^{2\pi i n x}\, d\,(P(x)1, 1),\end{aligned}$$

thus we can investigate the properties of the sequence $\int_0^1 \exp(\pi i S_{n;I}(x))dx$ by classifying the spectral types of the spectral measure P.

THEOREM 1. *Let P be the spectral measure corresponding to the unitary operator*

$$(Uf)(x) = \exp(\pi i \chi_I(x)) f(x+\theta), \quad f \in L^2(\mathbb{T}).$$

Then one and only one of the following three cases occurs:

(i) *P is absolutely continuous (hence Lebesgue), and*

$$\lim_{n\to\infty} \int_0^1 \exp(\pi i S_{n;I}(x))\,dx = 0.$$

(ii) *P is singular continuous, and for some set $J \subset \mathbb{N}$ of density zero,*

$$\lim_{n\notin J, n\to\infty} \int_0^1 \exp(\pi i S_{n;I}(x))\,dx = 0.$$

(iii) *P is discrete. In this case we have $\exp(\pi i \chi_I(x)) = \lambda \overline{g(x)} g(x+\theta)$ for some λ, g such that $|\lambda| = 1$, $|g(x)| \equiv 1$ a.e., and*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \left| \int_0^1 \exp(\pi i S_{n;I}(x))\,dx \right|^2 = \sum_{k=-\infty}^{\infty} |a_k|^4$$

where a_k is the k-th Fourier coefficient of $g(x)$.

Proof. To prove (i) is obvious we note that $\int_0^1 \exp(\pi i S_{n;I}(x))\,dx$ is the Fourier-Stieltjes coefficient of the absolutely continuous measure $E \mapsto$

$(P(E)1, 1)$, hence the Riemann-Lebesgue lemma implies the convergence to zero as $n \to \infty$.

Now N. Wiener's theorem says that for a Borel measure μ on $\mathbb{T}$,

$$\lim_{N\to\infty} \frac{1}{2N+1} \sum_{-N}^{N} |\widehat{\mu}(n)|^2 = \sum_{k=-\infty}^{\infty} |\mu(\{x_k\})|^2$$

where $\{x_k\}_k \subset \mathbb{T}$ are all the atoms of the measure μ and $\widehat{\mu}(n)$ is the n-th Fourier-Stieltjes coefficient of μ. So if P is continuous, then the measure $\mu_1(\cdot) = (P(\cdot)1, 1)$ is also continuous, hence the right-hand side is 0. Since

$$\overline{\widehat{\mu}_1(n)} = \widehat{\mu}_1(-n) = \int_0^1 e^{2\pi i n x} d(P(x)1,1) = (U^n 1, 1) = \int_0^1 \exp(\pi i S_n(x))\, dx,$$

we have $\lim_{N\to\infty} \frac{1}{N} \sum_1^N |\int_0^1 \exp(\pi i S_n(x))\, dx|^2 = 0$. Thus (ii) is proved.

Now we prove Part (iii). A unitary operator V defined by $(Vf)(x) = \overline{g(x)}g(x+\theta)f(x+\theta)$ for $f \in L^2(\mathbb{T})$ has the same spectral type as U and $|(U^n 1, 1)| = |(V^n 1, 1)|$ for every n since the ergodicity of irrational rotations implies $|\lambda| = 1$. Note that $MV = T_\theta M$ where M denotes the unitary operator defined by multiplication by the function g and T_θ is the rotation by θ. Hence V and T_θ are unitarily equivalent and $V^n = M^{-1} T_\theta^n M$. Note that

$$(V^n 1, 1) = (M^{-1} T_\theta^n M1, 1) = (T_\theta^n M1, M1) = (T_\theta^n g, g).$$

If P_k denotes the orthogonal projection in $L^2(\mathbb{T})$ onto the 1-dimensional subspace spanned by $e^{2\pi i k x}$ for each k, then $T_\theta = \sum_k e^{2\pi i k\theta} P_k$. If a spectral measure Q is defined by $Q(\{k\theta\}) = P_k$ for each k and $Q(E) = 0$ if $E \cap \{k\theta : k \in \mathbb{Z}\} = \emptyset$, then we have the spectral representation $T_\theta = \int_0^1 e^{2\pi i x} dQ(x)$. Since $g(x) = \sum_k a_k e^{2\pi i k x}$ and $Q(E)g = \sum_{k\theta\in E} P_k g = \sum_{k\theta \in E} a_k e^{2\pi i k x}$, we have

$$(Q(E)g, g) = (Q(E)g, Q(E)g) = \sum_{k\theta\in E} |a_k|^2 = \sum_k |a_k|^2\, \delta_{k\theta}(E),$$

where δ_τ is a Dirac mass at $\tau \in \mathbb{T}$. Now applying Wiener's theorem to the measure $(Q(\cdot)g, g)$ and using the fact that

$$|(V^{-n}1,1)| = |(V^n 1,1)| = |(T_\theta^n g, g)| = |\int_0^1 e^{2\pi i n x} d(Q(x)g,g)|,$$

we obtain $\lim_{N\to\infty} \frac{1}{N}\sum_1^N |(V^n 1,1)|^2 = \sum_{-\infty}^{\infty} |a_k|^4$. This completes the proof.

REMARK. If $A(x) = \exp(\pi i \chi_{[1-\theta,1)}(x))$, then we have $A(x) = e^{-\pi i\theta}$ $\overline{q(x)}q(x+\theta)$ where $q(x) = e^{\pi i x}$, $0 < x < 1$. In this case $a_k(q) = \frac{2}{\pi}\cdot\frac{1}{2k-1}$, hence $\sum_{k=-\infty}^{\infty} |a_k|^4 = \frac{1}{3}$. Note that we obtain the same value for any interval of length θ.

Now we define a subgroup G_θ of $\mathbb{T}$ as follows: $t \in G_\theta$ if and only if $\exp(\pi i \chi_{[0,t)})$ is a constant multiple of a coboundary. It can be shown that G_θ is measurable. Consider the functions $A_r(x)$ on $\mathbb{T}$ defined by $A_r(x) = \exp(\pi i \chi_{[0,r)})$, $0 < r < 1$, r rational. J.P. Conze [4] showed that $A_r(x)$ is not a constant multiple of a coboundary. Hence $r \notin G_\theta$ for such a rational number r. Note that the cosets of G_θ in $\mathbb{T}$ with representatives r rational, $0 < r < \frac{1}{2}$, are all distinct. Hence the index of the subgroup $G_\theta \subset \mathbb{T}$ is the continuum. For, if it were countable, then the total measure of the circle group would be zero. Hence the case (iii) in Theorem 1 hardly occurs.

If θ has bounded partial quotients, then $\mathbb{Z}\cdot\theta = G_\theta$. But this is not the case when θ has unbounded partial quotients. If θ be an irrational number with unbounded partial quotients, then the set G_θ is uncountable. (See [16]. And for related results, see [14], [15].)

For $x \in \mathbb{R}$, put $||x|| = \text{dist}(x, \mathbb{Z}) \le \frac{1}{2}$. The following result tells when the case (i) does not occur.

THEOREM 2. *Let θ be an irrational number with unbounded partial quotients and let $\{p_n/q_n\}_{n=1}^{\infty}$ be the convergents in its continued fraction expansion. If an interval $I \subset \mathbb{T}$ of length b satisfies the condition that $||q_n b|| \le C < \frac{1}{2}$ for every n except for finitely many n's where C is a constant, then the operator defined by*

$$(Uf)(x) = \exp(\pi i \chi_I(x)) f(x+\theta), \quad f \in L^2(\mathbb{T})$$

has a singular spectrum.

Proof. Since θ has unbounded partial quotients in its continued fraction expansion, we see that for any $\epsilon > 0$ there are infinitely many pairs of (p, q) such that $|\theta - p/q| \leq \epsilon \cdot 1/q^2$. In the rest of the proof, we assume that ϵ is sufficiently small, for example, $\epsilon < \frac{1}{2}(\frac{1}{2} - C)$. Note that such numbers p/q are necessarily convergents of θ. So we can choose an increasing sequence $\{q_{n_k}\}_{k=1}^{\infty}$ consisting of denominators of convergents of θ, which will be also denoted by $\{q_n\}_1^{\infty}$ for the sake of notational simplicity, such that for each n there exists an integer p_n, $(p_n, q_n) = 1$, satisfying $|\theta - (p_n/q_n)| \leq \epsilon \cdot 1/q_n^2$ and $||q_n b|| \leq C < \frac{1}{2}$.

Note that

$$\left|k\theta - \frac{kp_n}{q_n}\right| \leq \epsilon \cdot \frac{k}{q_n^2} < \epsilon \cdot \frac{1}{q_n}, \quad 0 \leq k < q_n.$$

Since $\{kp_n : 0 \leq k < q_n\} = \{0, 1, \dots, q_n - 1\} \pmod{q_n}$, we see that $\{kp_n/q_n : 0 \leq k < q_n\} = \{0, 1/q_n, 2/q_n, \dots, (q_n - 1)/q_n\}$ in $\mathbb{T}$ and $\{k\theta : 0 \leq k \leq q_n - 1\}$ are close to the points $\{0, 1/q_n, 2/q_n, \dots, (q_n - 1)/q_n\}$. More precisely, each open neighborhood of radius $\epsilon \cdot 1/q_n$ around the points $\{k \cdot 1/q_n : k = 0, \dots, q_n - 1\}$ contains one and only one point from the set $\{k\theta : k = 0, 1, \dots, q_n - 1\}$.

From now on we fix q_n. Let δ_n be the number satisfying $q_n \delta_n = ||q_n b||$. Then $0 \leq \delta_n \leq C \cdot (1/q_n)$ and $b = r \cdot (1/q_n) + \delta_n$ or $b = r \cdot (1/q_n) - \delta_n$ where r is the closest integer to $q_n b$.

Now we first consider the case when the interval I is of the form $I = [0, b)$. Put $I_1 = [0, r \cdot 1/q_n)$ and define two integer-valued functions on $\mathbb{T}$ by

$$S(x) \equiv S_{q_n;I}(x) = \mathrm{card}\{k : k\theta + x \in I, \quad 0 \leq k < q_n\}$$

and

$$S_1(x) \equiv S_{q_n;I_1}(x) = \mathrm{card}\{k : k\theta + x \in I_1, \quad 0 \leq k < q_n\}.$$

Then

$$|S(x) - S_1(x)| = S_{q_n;I_2} = \mathrm{card}\{k : k\theta + x \in I_2, \quad 0 \leq k < q_n\}$$

where I_2 is either one of the intervals of the form

$$[r/q_n, rq_n + \delta_n) \quad \text{or} \quad [r/q_n - \delta_n, r/q_n).$$

Note that any interval of length shorter than or equal to $1/(2q_n)$ cannot intersect more than one interval in $\mathbb{T}$ of the form

$$(k/q_n - \epsilon \cdot (1/q_n), k/q_n + \epsilon \cdot (1/q_n)), \quad 0 \le k < q_n,$$

since $\epsilon < 1/4$. Hence a translate of the interval I_2 contains at most one point from the set $\{k\theta : 0 \le k < q_n\}$ since I_2 has length equal to $\delta_n \le C \cdot (1/q_n) < \frac{1}{2} \cdot (1/q_n)$. That is, $|J(x) - J_1(x)| = 0$ or 1. Put $E = \bigcup_{k=0}^{q_n-1}(I_2 - k\theta)$. Then $E = \{x \in \mathbb{T} : |S(x) - S_1(x)| = 1\}$. Note that the measure of E does not exceed C and that $S(x) = S_1(x)$ on $\mathbb{T}\backslash E$. And put $F = \{x \in \mathbb{T} : S(x) = r\}$. Since $S_1(x) = r$ on the set

$$\mathbb{T}\backslash \bigcup_{k=0}^{q_n-1} \left[k \cdot \frac{1}{q_n} - \epsilon \cdot \frac{1}{q_n}, k \cdot \frac{1}{q_n} + \epsilon \cdot \frac{1}{q_n}\right),$$

we have that

$$F \supset \mathbb{T}\backslash\Big(\bigcup_{k=0}^{q_n-1} \left[k \cdot \frac{1}{q_n} - \epsilon \cdot \frac{1}{q_n}, k \cdot \frac{1}{q_n} + \epsilon \cdot \frac{1}{q_n}\right) \cup E\Big).$$

Note that $\exp(\pi i S(x)) = \pm e^{\pi i r}$ since $S(x)$ is an integer for every x. Put $F_1 = \{x : \exp(\pi i S(x)) = e^{\pi i r}\}$. Then $F \subset F_1$ and

$$m(F_1) \ge m(F) \ge 1 - (2\epsilon + m(E)) \ge 1 - (2\epsilon + C) = 1 - C - 2\epsilon > \frac{1}{2}.$$

Since

$$\begin{aligned}(U^{q_n}1, 1) &= \int_0^1 \exp(\pi i \sum_{k=0}^{q_n-1} \chi_I(x + k\theta))\, dx \\ &= \int_0^1 \exp(\pi i S_{q_n;I}(x)) dx \\ &= e^{\pi i r} \cdot m(F_1) - e^{\pi i r} \cdot (1 - m(F_1)) \\ &= e^{\pi i r}(2m(F_1) - 1),\end{aligned}$$

we have

$$|(U^{q_n}1,1)| = 2\cdot m(A_1) - 1 \geq 1 - 2C - 4\epsilon > 0.$$

This is true for infinitely many q_n. Therefore, $|(U^{q_n}1,1)| \geq K > 0$ for some constant K for every n, hence U cannot have absolutely continuous spectrum.

In the case that the interval I is of the form $[t_0, b+t_0) = t_0 + [0,b)$ we use $S(x-t_0)$ and $S_1(x-t_0)$ and obtain the same estimate as above which yields the same conclusion.

REMARK. Let θ be an irrational number with unbounded partial quotients. If b is a rational number where $0 < b = c/d < 1$, $(c,d) = 1$, d odd, then U has singular continuous spectrum since $||q_n b|| \leq \frac{1}{2}\cdot(1-\frac{1}{d})$ and since if the length of the interval I is rational other than 0 or 1 then Conze's result[4] implies that the function $\exp(\pi i \chi_I(x))$ is not a constant multiple of a coboundary hence the spectrum of U is continuous.

2. Winding numbers and spectra

As before, we let θ be an irrational number and define a unitary operator U on $L^2(\mathbb{T})$ by $(Uh)(x) = A(x)h(x+\theta)$, $h \in L^2(\mathbb{T})$, where $A(x)$ is a function of modulus 1. Then there exists a spectral measure P on $\mathbb{T}$ such that $U^n = \int_0^1 e^{2\pi i n x} dP(x)$. In this section we classify the spectral types of P by the winding number of the map $A : \mathbb{T} \to \mathbb{T}$ for sufficiently smooth A. For example, if $A(x) = e^{2\pi i m x}$ for some nonzero integer m, then $U^n 1$ is a constant multiple of $e^{2\pi i m n x}$, hence $(U^n 1, 1) = 0$ if and only if $n \neq 0$, therefore $E \mapsto (P(E)1,1)$ is the Lebesgue measure on $\mathbb{T}$ and the maximal spectral type of U is Lebesgue.

H. Furstenberg[6] proved that if a continuous mapping $A : \mathbb{T} \to \mathbb{T}$ has nonzero winding number and if it satisfies that for all $x, x' \in \mathbb{T}$, $|A(x) - A(x')| < M|x - x'|$ for some M, then it is not a constant multiple of a coboundary. Hence in this case we have a continuous spectrum.

The following result is due to A.G. Kushnirenko[12]. It will be needed in the proof of Theorem 5 which is the main theorem. (He proved the result for $m = 1$ but the same is obviously true for any nonzero m.)

LEMMA. *Let m be a nonzero integer, and let $f(x)$ be a real-valued C^2-function on $\mathbb{T}$. Then we have: if $|f'(x) + m| > 0$ on $\mathbb{T}$, then the spectrum* *of V is Lebesgue for every θ where $(Vh)(x) = e^{2\pi i(f(x)+mx)}h(x+\theta)$ for $h \in L^2(\mathbb{T})$.*

Note that $f(x)$ can be regarded as a periodic function with period 2π on the real line. We shall obtain a result in which there is no growth condition on $f(x)$. The following will be used later.

LEMMA 3. *For any real trigonometric polynomial $p(x)$ defined on $\mathbb{T}$ there exists a real trigonometric polynomial $w(x)$ such that $p(x) - \int_0^1 p(x)\,dx = w(x+\theta) - w(x)$ where θ is an irrational number.*

Proof. For sufficiently large N, we have $p(x) - \int_0^1 p(x)\,dx = \sum_{0<|n|\le N} a_n e^{2\pi inx}$. Since θ is irrational, we can find b_n such that $a_n = b_n(e^{2\pi in\theta} - 1)$, $0 < |n| \le N$. Then $w_1(x) \equiv \sum_{0<|n|\le N} b_n e^{2\pi inx}$ satisfies $p(x) - \int_0^1 p(x)\,dx = w_1(x+\theta) - w_1(x)$. Hence $w(x)$ is obtained by taking the real part of $w_1(x)$.

The following fact is needed to prove Proposition 4. For the proof, see [12].

KOKSMA'S INEQUALITY. *Let $0 \le x_1 < x_2 < x_3 < \cdots < x_n \le 1$ be an increasing sequence of n real numbers, and $f(x)$ a function on $\mathbb{T}$ of bounded variation. Then*

$$\left|\frac{1}{n}\sum_{i=1}^{n} f(x_i) - \int_0^1 f(x)\,dx\right| \le D_n^*(\{x_i\}_{i=1}^n)\cdot \mathrm{Var}(f)$$

where the discrepancy function $D_n^*(\{x_i\}_{i=1}^n)$ is defined by

$$D_n^*(\{x_i\}_{i=1}^n) = \max_{1\le i\le n}\left\{\max\left\{\left|\frac{i-1}{n} - x_i\right|, \left|\frac{i}{n} - x_i\right|\right\}\right\}$$

and $\mathrm{Var}(f)$ is the total variation of the function f on the interval $[0,1]$.

PROPOSITION 4. *Let f be a real-valued C^1-function on $\mathbb{T}$. Then the unitary operator $(Uh)(x) = e^{2\pi if(x)}h(x+\theta)$, $h \in L^2(\mathbb{T})$, has singular spectrum, i.e., its spectral type is either discrete or singular continuous.*

Proof. Note that $\int_0^1 f'(x)\,dx = 0$. Choose a real trigonometric polynomial $p_1(x)$ defined on $\mathbb{T}$ such that $\max_{x\in\mathbb{T}} |f'(x) - p_1(x)| < \epsilon/2$ for a sufficiently small positive constant ϵ. Then

$$\left|\int_0^1 p_1(x)\,dx\right| = \left|\int_0^1 \left[f'(x) - p_1(x)\right] dx\right| \le \int_0^1 |f'(x) - p_1(x)|\,dx < \frac{\epsilon}{2}.$$

Put $p(x) = p_1(x) - \int_0^1 p_1(x)\,dx$. Then we have

$$\max_{x\in\mathbb{T}} |f'(x) - p(x)| \le \max_{x\in\mathbb{T}} |f'(x) - p_1(x)| + \left|\int_0^1 p_1(x)\,dx\right| < \epsilon.$$

Note that $\int_0^1 p(x)\,dx = 0$. Put $P(x) = \int_0^x p(t)\,dt$. Then $P(x)$ is also a real trigonometric polynomial on $\mathbb{T}$. Now by Lemma 3, there exists a real trigonometric polynomial $w(x)$ on $\mathbb{T}$ such that $P(x) = w(x+\theta) - w(x) + c$ where c is a suitable constant. Put $g(x) = f(x) - P(x)$. Then g is of class C^1 on $\mathbb{T}$ and $|g'(x)| = |f'(x) - p(x)| < \epsilon$. Note that for $q(x) = e^{2\pi i w(x)}$ we have

$$\begin{aligned} e^{2\pi i f(x)} &= e^{2\pi i (g(x)+P(x))} \\ &= \lambda e^{2\pi i g(x)} e^{2\pi i w(x+\theta)} e^{-2\pi i w(x)} \\ &= \lambda e^{2\pi i g(x)} q(x+\theta)\overline{q(x)}. \end{aligned}$$

where $\lambda = e^{2\pi i c}$ is a constant.

Hence the unitary operator $U : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ which is defined by

$$(Uh)(x) = e^{2\pi i f(x)} h(x+\theta) = \overline{q(x)} \lambda e^{2\pi i g(x)} (qh)(x+\theta)$$

is unitarily equivalent to the unitary operator $(Vh)(x) = e^{2\pi i g(x)} h(x+\theta)$ since $VM = MU$ where M is the unitary operator given by the multiplication by q. Note that $(V^n 1, 1) = \lambda^n \int \exp(2\pi i \sum_{k=0}^{n-1} g(x + k\theta))\,dx$, $n \ge 1$.

Since the inequality $|\theta - p_n/q_n| < \frac{1}{\sqrt{5}}(1/q_n^2)$ holds for infinitely many pairs of two relatively prime numbers (p_n, q_n), where p_n/q_n is necessarily a convergent in the continued fraction expansion of θ, we can choose an increasing sequence $\{q_n\}_1^\infty$ for which the aforementioned inequality holds true. For q_n fixed, we have $\{kp_n/q_n : 0 \le k < q_n\} = \{0, 1/q_n, 2/q_n, \ldots, (q_n - 1)/q_n\}$ as subsets of $\mathbb{T}$ and the inequality $|k\theta -$

$kp_n/q_n| < \frac{1}{\sqrt{5}}(1/q_n)$ implies that each open neighborhood of radius $\frac{1}{\sqrt{5}}(1/q_n)$ around the points $\{k \cdot 1/q_n : k = 0, \dots, q_n - 1\}$ contains one and only one point from the set $\{k\theta : k = 0, 1, \dots, q_n - 1\}$. Hence the discrepancy function $D^*_{q_n}$ in Koksma's inequality satisfies $D^*_{q_n}(\{k\theta\}_{k=0}^{q_n-1}) < (1 + \frac{1}{\sqrt{5}})(1/q_n)$.

Note that $\mathrm{Var}(g) = \int_0^1 |g'(x)|\,dx < \epsilon$. Put $g_t(x) \equiv g(t + x)$, then $\int_0^1 g(x)\,dx = \int_0^1 g_t(x)\,dx$, $\mathrm{Var}(g_t) = \mathrm{Var}(g) < \epsilon$ for any t and Koksma's inequality implies that for any fixed $t \in \mathbb{T}$,

$$\begin{aligned}\Big|\sum_{k=0}^{q_n-1} g(t+k\theta) - q_n \int_0^1 g(x)\,dx\Big| &= q_n \Big|\frac{1}{q_n}\sum_{k=0}^{q_n-1} g(t+k\theta) - \int_0^1 g(x)\,dx\Big| \\ &= q_n \Big|\frac{1}{q_n}\sum_{k=0}^{q_n-1} g_t(k\theta) - \int_0^1 g_t(x)\,dx\Big| \\ &\le q_n \cdot D^*_{q_n}(\{k\theta\}_{k=0}^{q_n-1}) \cdot \mathrm{Var}(g_t) \\ &= (1 + \frac{1}{\sqrt{5}}) \cdot \epsilon.\end{aligned}$$

Since we can choose g so that ϵ is as small as we want, we see that for every $x \in \mathbb{T}$, $|\exp(2\pi i \sum_{k=0}^{q_n-1} g(x + k\theta)) - \mu_n| < \epsilon_1$, $\mu_n = \exp(2\pi i q_n \int_0^1 g(x)\,dx)$, for some ϵ_1 sufficiently small and that $|\int_0^1 \exp(2\pi i \sum_{k=0}^{q_n-1} g(x+k\theta))\,dx - \mu_n| < \epsilon_1$. Hence $\lim_{n\to\infty} |(V^{q_n}1, 1)| \ge 1 - \epsilon_1 > 0$ for every n and we conclude that the spectral type of V is not absolutely continuous.

Now we are ready to state the main theorem.

THEOREM 5. *Let $A(x)$ be a C^2-map from $\mathbb{T}$ to $\mathbb{T}$ with a winding number m. Then the unitary operator $(Uh)(x) = A(x)h(x + \theta)$, $h \in L^2(\mathbb{T})$ has Lebesgue spectrum if and only if $m \neq 0$.*

Proof. : If $A(x)$ has its winding number zero, then Proposition 4 implies that U has singular spectrum. Suppose $A(x)$ have a nonzero winding number m. In other words, $A(x) = \lambda e^{2\pi i(mx + f(x))}$ where f is a C^2-function on $\mathbb{T}$. As in the proof of Proposition 4, we can find

$q(x)$ and $g(x)$ satisfying $e^{2\pi i f(x)} = e^{2\pi i g(x)} q(x+\theta)\overline{q(x)}$ and $|g'(x)| < \epsilon$ for sufficiently small ϵ. Hence Kushnirenko's result implies that $V : L^2(\mathbb{T}) \to L^2(\mathbb{T})$, $(Vh)(x) = e^{2\pi i(g(x)+mx)} h(x+\theta)$ has Lebesgue spectrum since $|g'(x) + m| > 0$ for every $x \in \mathbb{T}$. Since $M_q U = V M_q$, we see that U and V are of the same spectral type, therefore we conclude that U has Lebesgue spectrum.

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