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EXTREME POINTS OF $\mathcal{A}_{2n}^{(n)}$

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1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space. If $\mathcal{L}$ is a lattice of orthogonal projections acting on $\mathcal{H}$, then $Alg\mathcal{L}$ is the algebra of all bounded operators acting on $\mathcal{H}$ that leave invariant every orthogonal projections in $\mathcal{L}$. A subspace lattice $\mathcal{L}$ is a strongly closed lattice of orthogonal projections acting on $\mathcal{H}$, containing 0 and I. Dually, if $\mathcal{A}$ is a subalgebra of $\mathcal{B}(\mathcal{H})$, the algebra consisting of all bounded operators acting on $\mathcal{H}$, then $Lat\mathcal{A}$ is the lattice of all orthogonal projections invariant for each operator in $\mathcal{A}$. An algebra $\mathcal{A}$ is reflexive if $\mathcal{A} = AlgLat\mathcal{A}$ and a lattice $\mathcal{L}$ is reflexive if $\mathcal{L} = LatAlg\mathcal{L}$. A lattice $\mathcal{L}$ is commutative if each pair of projections in $\mathcal{L}$ commutes. We write $(\mathcal{A})_1$ for the unit ball of the algebra $\mathcal{A}$. If $x_1, x_2, \ldots, x_m$ are vectors in some Hilbert space, then $[x_1, x_2, \ldots, x_m]$ means the closed subspace generated by the vectors $x_1, x_2, \ldots, x_m$. Let A be in $\mathcal{B}(\mathcal{H})$ and let x be in $\mathcal{H}$. If $\|Ax\| = \|A\|\|x\|$, then x is said to be a maximal vector for A and $\mathbf{max}A$ is the set of all maximal vectors for A. An element A of a subalgebra $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$ is described as an extreme point of $\mathcal{A}$ if the only way in which it can be expressed as a convex combination $A = \lambda B + (1-\lambda)C$, with $0 \le \lambda \le 1$ and B, C in $\mathcal{A}$, is by taking $B = C = A$.

Let $\mathcal{H}$ be a n-dimensional complex Hilbert space with an orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ and let $\mathcal{L}_n$ be the lattice generated by $\{[e_1], [e_3], \ldots, [e_{n-1}], [e_1, e_2, e_3], [e_3, e_4, e_5], \ldots, [e_{n-3}, e_{n-2}, e_{n-1}], [e_{n-1}, e_n]\}$ if n is even(or. $\{[e_1], [e_3], \ldots, [e_n], [e_1, e_2, e_3], [e_3, e_4, e_5], \ldots, [e_{n-2}, e_{n-1}, e_n]\}$ if n is odd). Let $\mathcal{H}$ be an infinite separable Hilbert space with orthonormal basis $\{e_1, e_2, \ldots\}$ and let $\mathcal{L}_\infty$ be the lattice generated by $\{[e_{2i-1}], [e_{2i-1}, e_{2i}, e_{2i+1}] : i = 1, 2, \ldots\}$. Then the extreme points of the algebras $Alg\mathcal{L}_{2n}$ and $Alg\mathcal{L}_\infty$ are investigated in [8].

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Let $\mathcal{H}$ be a $2n$-dimensional complex Hilbert space with an orthonormal basis $\{e_1, e_2, \ldots, e_{2n}\}$. Let $\mathcal{L}_{2n}$ be the lattice generated by $\{[e_1], [e_3], \ldots, [e_{2n-1}], [e_1, e_2, e_3], [e_3, e_4, e_5], \ldots, [e_1, e_{2n-1}, e_{2n}]\}$ and let $\mathcal{A}_{2n}$ be the tridiagonal algebra discovered by F. Gilfeather and D. Larson. Then $\mathcal{A}_{2n} = Alg\mathcal{L}_{2n}$ and $A \in \mathcal{A}_{2n}$ has the form

$$\begin{pmatrix} * & * & & & & * \\ & * & & & & \\ & * & * & * & & \\ & & & * & & \\ & & & * & \ddots & \\ & & & & \ddots & * \\ & & & & & * \end{pmatrix},$$

with respect to the basis $\{e_1, e_2, \ldots, e_{2n}\}$, where all non-starred entries are zero. The extreme points of $\mathcal{A}_{2n}$ are investigated in [2]. If we write the basis in the order $\{e_1, e_3, \ldots, e_{2n-1}, e_2, e_4, \ldots, e_{2n}\}$, then the above matrix looks like this

$$\begin{pmatrix} * & & & & & * & & & & * \\ & * & & & & * & * & & & \\ & & \ddots & & & & * & & & \\ & & & \ddots & & & & \ddots & & \\ & & & & \ddots & & & & \ddots & \\ & & & & * & & & & * & * \\ & & & & & * & & & & \\ & & & & & & \ddots & & & \\ & & & & & & & \ddots & & \\ & & & & & & & & \ddots & \\ & & & & & & & & & * \end{pmatrix},$$

where all non-starred entries are zero.

Let $\mathcal{A}_{2n}^{(n)} = \{\begin{pmatrix} D_1 & S \\ 0 & D_2 \end{pmatrix} : D_1$ and D_2 are $n \times n$ diagonal matrices and S is an $n \times n$ matrix $\}$. The isometries of $\mathcal{A}_{2n}^{(n)}$ are investigated in [7]. In this paper we will investigate the extreme points of $\mathcal{A}_{2n}^{(n)}$.

2. Preliminaries and general properties

LEMMA 2.1 [8]. *Let $\mathcal{H}$ be a finite dimensional Hilbert space and let A be in $\mathcal{B}(\mathcal{H})$ such that $\|A\| = 1$. Then A has at least one nonzero maximal vector.*

LEMMA 2.2 [8]. *Let $\mathcal{A}$ be a subalgebra of $\mathcal{B}(\mathcal{H})$. If A is an extreme point of $(\mathcal{A})_1$, then A^* is an extreme point of $(\mathcal{A}^*)_1$, where $\mathcal{A}^* = \{A^* : A \in \mathcal{A}\}$.*

LEMMA 2.3 [8]. *Let A be a nonzero operator in $\mathcal{B}(\mathcal{H})$. Then*

$$\mathbf{max}A = \mathbf{ker}(\|A\|^2 - A^*A) \text{ and } \dim(\mathbf{max}A) = \dim(\mathbf{max}A^*).$$

LEMMA 2.4 [8]. *Let $\dim \mathcal{H} < \infty$ and let $P \geq 0$. Then $\mathbf{ran}P = \mathbf{ran}P^{\frac{1}{2}}$, where $\mathbf{ran}P = \{Px : x \in \mathcal{H}\}$.*

LEMMA 2.5 [11]. *Let $\mathcal{L}$ be a nest or a distributive lattice of orthogonal projections. If A is in $(Alg\mathcal{L})_1$, then A is an extreme point of $(Alg\mathcal{L})_1$ if and only if for all E in $\mathcal{L}$, either $E \cap \mathbf{ran}(1 - AA^*)^{\frac{1}{2}} = \{0\}$ or $E_-^\perp \cap \mathbf{ran}(1 - A^*A)^{\frac{1}{2}} = \{0\}$, where $E_- = \vee\{F : F \in \mathcal{L} \text{ and } F \not\geq E\}$ and $E_-^\perp = (E_-)^\perp$.*

LEMMA 2.6 [8]. *Let A be in $\mathcal{A}_{2n}$ such that $\|A\| = 1$. Then A is an extreme point of $(\mathcal{A}_{2n})_1$ if and only if for all E in $\mathcal{L}_{2n}$, either $\mathbf{max}A^* \vee E^\perp = \mathcal{H}$ or $\mathbf{max}A \vee E_- = \mathcal{H}$.*

LEMMA 2.7 [2]. *Let $\mathcal{A}$ and $\mathcal{B}$ be subalgebras of $\mathcal{B}(\mathcal{H})$. Let U be a unitary operator such that $U\mathcal{A}U^* = \mathcal{B}$. Then A is an extreme point of $(\mathcal{A})_1$ if and only if UAU^* is an extreme point of $(\mathcal{B})_1$.*

Proof. If $UAU^* = \lambda B + (1-\lambda)C$ for some B and C in $(\mathcal{B})_1$, then $A = \lambda U^*BU + (1-\lambda)U^*CU$ and U^*BU and U^*CU are in $(\mathcal{A})_1$. Since A is an extreme point of $(\mathcal{A})_1$, $A = U^*BU = U^*CU$. Since U is unitary, $UAU^* = B = C$. Conversely, if $A = \lambda B + (1-\lambda)C$ for some B and C in $(\mathcal{A})_1$, then $UAU^* = \lambda UBU^* + (1-\lambda)UCU^*$ and UBU^* and UCU^* are in $(\mathcal{B})_1$. Since UBU^* is an extreme point of $(\mathcal{B})_1$, $UAU^* = UBU^* = UCU^*$. Hence $A = B = C$.

LEMMA 2.8 [2]. *Let A and U be in $\mathcal{B}(\mathcal{H})$ and let U be a unitary operator. Then $U^*(\mathbf{max}A) = \mathbf{max}(U^*AU)$.*

Proof. Let x be in $\mathbf{max}A$. Then $\|Ax\| = \|A\|\|x\|$ and so $\|U^*AUU^*x\| = \|U^*Ax\| = \|Ax\| = \|A\|\|x\| = \|U^*AU\|\|U^*x\|$. Hence U^*x is in $\mathbf{max}(U^*AU)$. Conversely, let x be in $\mathbf{max}(U^*AU)$. Then $\|AUx\| = \|U^*AUx\| = \|U^*AU\|\|x\| = \|A\|\|x\| = \|A\|\|Ux\|$. Hence Ux is in $\mathbf{max}A$.

THEOREM 2.9. *Let A and U be in $\mathcal{B}(\mathcal{H})$ and let U be a unitary operator. Then $\dim(\mathbf{max}(UAU^*)) = \dim(\mathbf{max}A)$.*

3. Extreme point of $\mathcal{A}_{2n}^{(n)}$

Let $\mathcal{H}$ be a $2n$-dimensional complex Hilbert space with an orthonormal basis $\{e_1, e_2, \ldots, e_{2n}\}$ and let $\mathcal{L}_{2n}^{(n)}$ be the lattice generated by $\{[e_1], [e_2], \ldots, [e_n], [e_1, e_2, \ldots, e_n, e_{n+i}] : i = 1, 2, \ldots, n\}$. Then $\mathcal{A}_{2n}^{(n)} = Alg\mathcal{L}_{2n}^{(n)}$ and $\mathcal{A}_{2n}^{(n)}$ and $\mathcal{L}_{2n}^{(n)}$ are reflexive.

First we consider the extreme points of $\mathcal{A}_2^{(1)}$ and $\mathcal{A}_4^{(2)}$. Since $\mathcal{A}_2^{(1)} = \mathcal{A}_2$, we have the following theorem.

THEOREM 3.1 [2]. *Let $A = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}$ be in $\mathcal{A}_2^{(1)}$ such that $\|A\| = 1$. Then A is an extreme point of $(\mathcal{A}_2^{(1)})_1$ unless A is diagonal such that $|a_{ii}| < 1$ for some i $(i = 1, 2)$.*

Let $A = \begin{pmatrix} D_1 & S \\ 0 & D_2 \end{pmatrix} \in \mathcal{A}_4^{(2)}$ such that $\|A\| = 1$, where

$$D_1 = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix}, D_2 = \begin{pmatrix} a_{33} & 0 \\ 0 & a_{44} \end{pmatrix} \text{ and } S = \begin{pmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{pmatrix}.$$

Let U be a 4×4 unitary matrix with 1 in $(1,1)$-, $(2,3)$-, $(3,2)$-,and $(4,4)$-components and 0 elsewhere. Then $U = U^*$ and $U\mathcal{A}_4U^* = \mathcal{A}_4^{(2)}$. By Lemma 2.8, A is an extreme point of $(\mathcal{A}_4^{(2)})_1$ if and only if UAU^* is an extreme point of $(\mathcal{A}_4)_1$. Using the result in [2] we have the following theorems.

THEOREM 3.2. *Let A be in $\mathcal{A}_4^{(2)}$ such that $\|A\| = 1$. If every entry of S is nonzero, then A is an extreme point of $(\mathcal{A}_4^{(2)})_1$ if and only if* $\dim(\mathbf{max}A) = 2$.

THEOREM 3.3. *Let $A = \begin{pmatrix} D_1 & S \\ 0 & D_2 \end{pmatrix}$ be in $\mathcal{A}_4^{(2)}$ such that $\|A\| = 1$. If exactly one element of S is zero, then A is not an extreme point of $(\mathcal{A}_4^{(2)})_1$.*

THEOREM 3.4. *Let A be in $\mathcal{A}_4^{(2)}$ such that $\|A\| = 1$ and let exactly k $(2 \leq k \leq 4)$ elements of S be zero. Then A is an extreme point of $(\mathcal{A}_4^{(2)})_1$ if and only if* $\dim(\mathbf{max}A) = k$.

From now on, we will consider the extreme points of $(\mathcal{A}_{2n}^{(n)})_1$ for all positive integers n. Let $A = \begin{pmatrix} D_1 & S \\ \mathbf{0} & D_2 \end{pmatrix}$ be in $\mathcal{A}_{2n}^{(n)}$ such that $\|A\| = 1$, where D_1 is an $n \times n$ diagonal matrix with a_{ii} in (i,i)-components, D_2 is an $n \times n$ diagonal matrix with $a_{n+j,n+j}$ in (j,j)-components and S is an $n \times n$ matrix with $a_{i,n+j}$ in (i,j)-components for all i,j $(1 \leq i,j \leq n)$. Then

$$A^*A = \begin{pmatrix} D_1^*D_1 & D_1^*S \\ S^*D_1 & S^*S + D_2^*D_2 \end{pmatrix}.$$

If $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2) = (x_1, x_2, \ldots, x_{2n}) \in \mathbf{max}A$, where $\mathbf{x}_1 = (x_1, x_2, \ldots, x_n)$ and $\mathbf{x}_2 = (x_{n+1}, x_{n+2}, \ldots, x_{2n})$, then by Lemma 2.3, we have the following equation

$$\begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} = \begin{pmatrix} D_1^*D_1 & D_1^*S \\ S^*D_1 & S^*S + D_2^*D_2 \end{pmatrix} \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix}.$$

From this we have the following relations;

$$(*) \qquad \alpha_1 x_1 = \sum_{k=1}^{n} \overline{a}_{11} a_{1,n+k} x_{n+k}$$

$$\alpha_2 x_2 = \sum_{k=1}^{n} \overline{a}_{22} a_{2,n+k} x_{n+k}$$

$$\vdots$$

$$\alpha_n x_n = \sum_{k=1}^{n} \overline{a}_{nn} a_{n,n+k} x_{n+k}$$

$$\alpha_{n+1} x_{n+1} = \sum_{k=1}^{n} \overline{a}_{k,n+1} a_{kk} x_k + \sum_{j=2}^{n} (\sum_{k=1}^{n} \overline{a}_{k,n+1} a_{k,n+j}) x_{n+j}$$

$$\alpha_{n+2} x_{n+2} = \sum_{k=1}^{n} \overline{a}_{k,n+2} a_{kk} x_k + \sum_{j\neq 2, j=1}^{n} (\sum_{k=1}^{n} \overline{a}_{k,n+2} a_{k,n+j}) x_{n+j}$$

$$\vdots$$

$$\alpha_{2n} x_{2n} = \sum_{k=1}^{n} \overline{a}_{k,2n} a_{kk} x_k + \sum_{j=1}^{n-1} (\sum_{k=1}^{n} \overline{a}_{k,2n} a_{k,n+j}) x_{n+j}$$

where $\alpha_i = 1 - |a_{ii}|^2$, and $\alpha_{n+i} = 1 - |a_{n+i,n+i}|^2 - \sum_{k=1}^{n} |a_{k,n+i}|^2$ for all $i = 1, 2, \ldots, n$.

From this relations we have the following theorem.

THEOREM 3.5. *Let* $A = \begin{pmatrix} D_1 & S \\ 0 & D_2 \end{pmatrix}$ *be in* $\mathcal{A}_{2n}^{(n)}$ *such that* $\|A\| = 1$. *If each row vector of* S *has at least one nonzero element, then* $\dim(\mathbf{max}A) \leq n$.

COROLLARY 3.6. *Let* $A = \begin{pmatrix} D_1 & S \\ 0 & D_2 \end{pmatrix}$ *be in* $\mathcal{A}_{2n}^{(n)}$ *such that* $\|A\| = 1$. *If each column vector of* S *has at least one nonzero element, then* $\dim(\mathbf{max}A^*) \leq n$.

THEOREM 3.7. *Let* A *be in* $\mathcal{A}_{2n}^{(n)}$ *such that* $\|A\| = 1$. *If* A *is an extreme point of* $(\mathcal{A}_{2n}^{(n)})_1$, *then* $\dim(\mathbf{max}A) \geq n$.

Proof. Suppose that $\dim(\mathbf{max}A) = k < n$. Take $E = [e_1, e_2, \ldots, e_{k+1}]$ in $\mathcal{L}_{2n}^{(n)}$. Then $E_- = [e_1, e_2, \ldots, e_n]$ and $E^\perp = [e_1, e_2, \ldots, e_{k+1}]^\perp$. Hence $\dim(E_-) = n$ and $\dim(E^\perp) = 2n - (k+1)$. Thus $\mathbf{max}A \vee E_- \neq \mathcal{H}$ and $\mathbf{max}A^* \vee E^\perp \neq \mathcal{H}$.

COROLLARY 3.8. *Let* $A = \begin{pmatrix} D_1 & S \\ 0 & D_2 \end{pmatrix}$ *be in* $\mathcal{A}_{2n}^{(n)}$ *such that* $\|A\| = 1$ *and let each row vector of* S *be nonzero or each column vector of* S *be nonzero. If* A *is an extreme point of* $(\mathcal{A}_{2n}^{(n)})_1$, *then* $\dim(\mathbf{max}A) = n$.

COROLLARY 3.9. *Let* $A = \begin{pmatrix} D_1 & S \\ 0 & D_2 \end{pmatrix}$ *be in* $\mathcal{A}_{2n}^{(n)}$ *such that* $\|A\| = 1$ *and each row vector of* S *has at least one nonzero element. If for some* i $(1 \le i \le n)$, $x_{n+i} = 0$ *for all* $\mathbf{x}=(x_1, x_2, \ldots, x_{2n}) \in \mathbf{max}A$, *then* A *is not an extreme point of* $(\mathcal{A}_{2n}^{(n)})_1$.

Proof. From the equation $(*)$, $\dim(\mathbf{max}A) \le n-1$. Hence by Theorem 3.7, A is not extreme.

COROLLARY 3.10. *Let* $A = \begin{pmatrix} D_1 & S \\ 0 & D_2 \end{pmatrix}$ *be in* $\mathcal{A}_{2n}^{(n)}$ *such that* $\|A\| = 1$ *and each column vector of* S *has at least one nonzero element. If for some* i $(1 \le i \le n)$, $y_i = 0$ *for all* $\mathbf{y}=(y_1, y_2, \ldots, y_{2n}) \in \mathbf{max}A^*$, *then* A *is not an extreme point of* $(\mathcal{A}_{2n}^{(n)})_1$.

THEOREM 3.11. *Let* A *be in* $\mathcal{A}_{2n}^{(n)}$ *such that* $\|A\| = 1$. *If* A *is an extreme point of* $(\mathcal{A}_{2n}^{(n)})_1$, *then for each* i $(1 \le i \le n)$, *either there exists* $\mathbf{x}_{n+i} = (x_1, x_2, \ldots, x_{2n}) \in \mathbf{max}A$ *such that* $x_{n+i} = 1$ *and* $x_{n+j} = 0$ *for all* j $(j \ne i, 1 \le j \le n)$ *or there exists* $\mathbf{y}_i = (y_1, y_2, \ldots, y_{2n}) \in \mathbf{max}A^*$ *such that* $y_i = 1$ *and* $y_j = 0$ *for all* j $(j \ne i, 1 \le j \le n)$.

Proof. Let $E = [e_1, e_2, \ldots, e_n]$. Then $E_- = [e_1, e_2, \ldots, e_n]$ and $E^\perp = [e_{n+1}, e_{n+2}, \ldots, e_{2n}]$. Since A is an extreme point of $(\mathcal{A}_{2n}^{(n)})_1$, either $\mathbf{max}A \vee E_- = \mathcal{H}$ or $\mathbf{max}A \vee E^\perp = \mathcal{H}$. If $\mathbf{max}A \vee E_- = \mathcal{H}$, then $e_{n+i} \in \mathbf{max}A \vee E_-$ for all $i = 1, 2, \ldots, n$. So $e_{n+i} = \sum_{k=1}^{2n} x_k e_k + \sum_{k=1}^{n} \mu_k e_k$ for some $\sum_{k=1}^{2n} x_k e_k \in \mathbf{max}A$ and $\sum_{k=1}^{n} \mu_k e_k \in E_-$. Hence $x_{n+i} = 1$ and $x_{n+j} = 0$ for all j $(j \ne i, 1 \le j \le n)$. Thus there exists $\mathbf{x}_i = (x_1, x_2, \ldots, x_{2n}) \in \mathbf{max}A$ such that $x_{n+i} = 1$ and $x_{n+j} = 0$ for all j $(j \ne i, 1 \le j \le n)$. Similarly, if $\mathbf{max}A \vee E^\perp = \mathcal{H}$, then there exists $\mathbf{y}_i = (y_1, y_2, \ldots, y_{2n}) \in \mathbf{max}A^*$ such that $y_i = 1$ and $y_j = 0$ for all j $(j \ne i, 1 \le j \le n)$.

THEOREM 3.12. *Let* A *be in* $\mathcal{A}_{2n}^{(n)}$ *such that* $\|A\| = 1$. *If for each* i $(1 \le i \le n)$, *there exists a vector* $\mathbf{x}= (x_1, x_2, \ldots, x_{2n}) \in \mathbf{max}A$ *such* *that $x_{n+i} \neq 0$ and $x_{n+j} = 0$ for all j ($j \neq i, 1 \leq j \leq n$), and there exists a vector $\mathbf{y}=(y_1, y_2, \ldots, y_{2n}) \in \mathbf{max}A^*$ such that $y_i \neq 0$, then A is an extreme point of $(\mathcal{A}_{2n}^{(n)})_1$.*

Proof. Let $E = [e_k]$ for $k = 1, 2, \ldots, n$. Then $E^\perp = [e_k]^\perp$ and so $\mathbf{max}A^* \vee E^\perp = \mathcal{H}$. Let $E = [e_1, e_2, \ldots, e_n, e_{n+k}]$ for some $k = 1, 2, \ldots, n$. Then $E_- = [e_{n+k}]^\perp$ and so $\mathbf{max}A \vee E_- = \mathcal{H}$. Let E be in $\mathcal{L}_{2n}^{(n)}$ such that $[e_j] \subsetneq E \subset [e_1, e_2, \ldots, e_n]$ for some j ($1 \leq j \leq n$). Then $E_- = [e_1, e_2, \ldots, e_n]$. Since $e_{n+k} \in \mathbf{max}A \vee E_-$ for all $k = 1, 2, \ldots, n$, $\mathbf{max}A \vee E_- = \mathcal{H}$. Let E be in $\mathcal{L}_{2n}^{(n)}$ such that $[e_1, e_2, \ldots, e_n, e_{n+j}] \subsetneq E$ for some j ($1 \leq j \leq n$). Then $E_- = \mathcal{H}$ and so $\mathbf{max}A \vee E_- = \mathcal{H}$. If $E \in \mathcal{L}_{2n}^{(n)}$ such that E is different from above cases, then $E_- = \mathcal{H}$ and so $\mathbf{max}A \vee E_- = \mathcal{H}$.

By an argument similar to Theorem 3.12, we can get the following theorem.

THEOREM 3.13. *Let A be in $\mathcal{A}_{2n}^{(n)}$ such that $\|A\| = 1$. If for each i ($1 \leq i \leq n$), there exists a vector $\mathbf{x}=(x_1, x_2, \ldots, x_{2n}) \in \mathbf{max}A$ such that $x_{n+i} \neq 0$ and there exists a vector $\mathbf{y}=(y_1, y_2, \ldots, y_{2n}) \in \mathbf{max}A^*$ such that $y_i \neq 0$ and $y_j = 0$ for all j ($j \neq i, 1 \leq j \leq n$), then A is an extreme point of $(\mathcal{A}_{2n}^{(n)})_1$.*

THEOREM 3.14. *Let $A = \begin{pmatrix} D_1 & S \\ 0 & D_2 \end{pmatrix}$ be in $\mathcal{A}_{2n}^{(n)}$ such that $\|A\| = 1$. If each row vector of S is nonzero and $\mathbf{x}_{n+k} \in \mathbf{max}A$ for all $k = 1, 2, \ldots, n$, where $\mathbf{x}_{n+k} = (x_1, x_2, \ldots, x_{2n})$ with $x_{n+k} = 1$ and $x_{n+j} = 0$ for all j ($j \neq k, 1 \leq j \leq n$), then A is an extreme point of $(\mathcal{A}_{2n}^{(n)})_1$.*

Proof. Let $D_1 = (a_{ii})$ and $D_2 = (a_{n+i,n+i})$ be $n \times n$ diagonal matrices and let $S = (a_{i,n+j})$ be $n \times n$ matrix. Suppose that $a_{1,n+p_1} \neq 0, a_{2,n+p_2} \neq 0, \ldots, a_{n,n+p_n} \neq 0$ ($1 \leq p_1, p_2, \ldots, p_n \leq n$). Then $|a_{ii}| \neq 1$ for all $i = 1, 2, \ldots, n$. Since $\mathbf{x}_{n+p_j} \in \mathbf{max}A$, $A\mathbf{x}_{n+p_j} \in \mathbf{max}A^*$ and the j-th component of $A\mathbf{x}_{n+p_j}$ is $a_{jj}x_j + a_{j,n+p_j}$. Since $x_j = \alpha_j^{-1}\overline{a}_{jj}a_{j,n+p_j}$, $a_{jj}x_j + a_{j,n+p_j} = a_{jj}\alpha_j^{-1}\overline{a}_{jj}a_{j,n+p_j} + a_{j,n+p_j} = a_{j,n+p_j}(\alpha_j^{-1}|a_{jj}|^2 + 1) = \alpha_j^{-1}a_{j,n+p_j} \neq 0$. Hence for each j ($1 \leq j \leq n$), there exist $\mathbf{y}_j=(y_1, y_2, \ldots, y_{2n}) \in \mathbf{max}A^*$ such that $y_j \neq 0$. By Theorem 3.12, A is an extreme point of $(\mathcal{A}_{2n}^{(n)})_1$.

By an argument similar to Theorem 3.14, we can get the following theorem.

THEOREM 3.15. *Let A be in $\mathcal{A}_{2n}^{(n)}$ such that $\|A\| = 1$. If each column vector of S is nonzero and $\mathbf{y}_k \in \mathbf{max}A^*$ for all $k = 1, 2, \ldots, n$, where $\mathbf{y}_k = (y_1, y_2, \ldots, y_{2n})$ with $y_k = 1$ and $y_j = 0$ for all j $(j \neq k, 1 \leq j \leq n)$, then A is an extreme point of $(\mathcal{A}_{2n}^{(n)})_1$.*

Let $A = \begin{pmatrix} D_1 & S \\ 0 & D_2 \end{pmatrix}$ be in $\mathcal{A}_{2n}^{(n)}$ such that $\|A\| = 1$. Let k $(1 \leq k \leq n)$ be given and let $\mathbf{x}_{n+k} = (x_1, x_2, \ldots, x_{2n})$ with $x_{n+k} = 1$ and $x_{n+j} = 0$ for all j $(j \neq k, 1 \leq j \leq n)$. Then $\mathbf{x}_{n+k} \in \mathbf{max}A$ if and only if $x_i = \alpha_i^{-1}\overline{a}_{ii}a_{i,n+k}$ for all i $(1 \leq i \leq n)$ provided $\alpha_i \neq 0$ and

$$\begin{pmatrix} \overline{a}_{1,n+1}a_{1,n+k} & \overline{a}_{2,n+1}a_{2,n+k} & \cdots & \overline{a}_{n,n+1}a_{n,n+k} \\ \overline{a}_{1,n+2}a_{1,n+k} & \overline{a}_{2,n+2}a_{2,n+k} & \cdots & \overline{a}_{n,n+2}a_{n.n+k} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \overline{a}_{1,2n}a_{1,n+k} & \overline{a}_{2,2n}a_{2,n+k} & \cdots & \overline{a}_{n,2n}a_{n,n+k} \end{pmatrix} \begin{pmatrix} \alpha_1^{-1} \\ \alpha_2^{-1} \\ \cdot \\ \cdot \\ \cdot \\ \alpha_n^{-1} \end{pmatrix} = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \cdot \\ \cdot \\ \cdot \\ \gamma_n \end{pmatrix}$$

where $\alpha_j = 1 - |a_{jj}|^2$ and $\alpha_j^{-1} = 0$ if $\alpha_j = 0$ for all $j = 1, 2, \ldots, n$ and $\gamma_k = 1 - |a_{n+k,n+k}|^2$ and $\gamma_j = 0$ for all j $(j \neq k, 1 \leq j \leq n)$. Suppose that $S_i^* = (\overline{a}_{i,n+1},\ \overline{a}_{i,n+2},\ \ldots,\ \overline{a}_{i,2n})^t$, that is, S_i^* is the i-th column vector of S^*, for all $i = 1, 2, \ldots, n$. Let $B = (\alpha_1^{-1}, \alpha_2^{-1}, \ldots, \alpha_n^{-1})^t$ and let $P_k = (\gamma_1, \gamma_2, \ldots, \gamma_n)^t$. Then the above equation holds if and only if

$$(a_{1,n+k}S_1^*,\ a_{2,n+k}S_2^*,\ \ldots,\ a_{n,n+k}S_n^*)B = P_k.$$

From this fact we have the following theorem.

THEOREM 3.16. *Let $A = \begin{pmatrix} D_1 & S \\ 0 & D_2 \end{pmatrix}$ be in $\mathcal{A}_{2n}^{(n)}$ such that $\|A\| = 1$. Let k $(1 \leq k \leq n)$ be given and let $\mathbf{x}_{n+k} = (x_1, x_2, \ldots, x_{2n})$ with $x_{n+k} = 1$ and $x_{n+j} = 0$ for all j $(j \neq k, 1 \leq j \leq n)$. Then $\mathbf{x}_{n+k} \in \mathbf{max}A$ if and only if*

$$(a_{1,n+k}S_1^*,\ a_{2,n+k}S_2^*,\ \ldots,\ a_{n,n+k}S_n^*)B = P_k$$

and $x_i = \alpha_i^{-1}\overline{a}_{ii}a_{i,n+k}$ *for all* i $(1 \le i \le n)$ *provided* $\alpha_i \neq 0$.

Let S_i be the ith-column vector of S and let $C = (\beta_1^{-1}, \beta_2^{-1}, \dots, \beta_n^{-1})^t$, where $\beta_i = 1 - |a_{n+i,n+i}|^2$ and $\beta_i^{-1} = 0$ if $\beta_i = 0$ for all $i = 1, 2, \dots, n$. Let $Q_k = (\eta_1, \eta_2, \dots, \eta_n)^t$, where $\eta_k = \alpha_k$ and $\eta_j = 0$ if $j \neq k$. By an argument similar to Theorem 3.16, we can get the following theorem.

THEOREM 3.17. *Let* $A = \begin{pmatrix} D_1 & S \\ 0 & D_2 \end{pmatrix}$ *be in* $\mathcal{A}_{2n}^{(n)}$ *such that* $\|A\| = 1$. *Let* k $(1 \le k \le n)$ *be given and let* $\mathbf{y}_k = (y_1, y_2, \dots, y_{2n})$ *with* $y_k = 1$ *and* $y_i = 0$ *for all* i $(i \neq k, 1 \le i \le n)$. *Then* $\mathbf{y}_k \in \mathbf{max}A^*$ *if and only if*

$$(\overline{a}_{k,n+1}S_1,\ \overline{a}_{k,n+2}S_2,\ \dots,\ \overline{a}_{k,2n}S_{2n})C = Q_k$$

and $y_{n+i} = \beta_i^{-1}\overline{a}_{k,n+i}a_{n+i,n+i}$ *for all* i $(1 \le i \le n)$ *such that* $\beta_i \neq 0$.

From Theorem 3.11, 3.12, 3.13, 3.16 and 17, we have the following theorems.

THEOREM 3.18. *Let* $A = \begin{pmatrix} D_1 & S \\ 0 & D_2 \end{pmatrix}$ *be in* $\mathcal{A}_{2n}^{(n)}$ *such that* $\|A\| = 1$. *If* A *is an extreme point of* $(\mathcal{A}_{2n}^{(n)})_1$, *then for each* k $(1 \le k \le n)$,

$$(a_{1,n+k}S_1^*,\ a_{2,n+k}S_2^*,\ \dots,\ a_{n,n+k}S_n^*)B = P_k$$

or

$$(a_{k,n+1}S_1,\ a_{k,n+2}S_2,\ \dots,\ a_{k,2n}S_n)C = Q_k.$$

THEOREM 3.19. *Let* $A = \begin{pmatrix} D_1 & S \\ 0 & D_2 \end{pmatrix}$ *be in* $\mathcal{A}_{2n}^{(n)}$ *such that* $\|A\| = 1$ *and let each row and column vector has at least one nonzero element. If for each* k $(1 \le k \le n)$,

$$(a_{1,n+k}S_1^*,\ a_{2,n+k}S_2^*,\ \dots,\ a_{n,n+k}S_n^*)B = P_k$$

or

$$(a_{k,n+1}S_1,\ a_{k,n+2}S_2,\ \dots,\ a_{k,2n}S_n)C = Q_k,$$

then A *is extreme.*

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