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ON EXTREME POINTS OF THE FAMILY OF SPIRALLIKE FUNCTIONS

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1. Introduction

Let A be the set of analytic functions on the open unit disk Δ. With the usual topology of uniform convergence on compact subsets of Δ, A is a locally convex linear topological space.

For $F \subset A$, the closed convex hull of F is defined as the intersection of all closed convex sets containing F. We use the notation $\overline{\text{co}}\, F$ for the closed convex hull of F. It is known [p.44, 7] that the convex hull of F consists of all elements of the form $\sum_{k=1}^{m} t_k f_k$ where $f_k \in F$, $t_k \geq 0$ and $\sum_{k=1}^{m} t_k = 1$.

Suppose that F is a compact subset of A. A function f is called an extreme point of F if $f \in F$ and if

$$f = tf_1 + (1-t)f_2 \text{ implies } f = f_1 = f_2$$

whenever $0 < t < 1$ and $f_1, f_2 \in F$. That is, an extremal subset of $F \subset A$ which consists of just one point is called an extreme point of F. We shall use the notation EF to denote the set of extreme points of F.

An α-spiral is a curve in the complex plane of the form

$$w = w_0 \exp(-e^{-i\alpha} t), \ -\infty < t < \infty, \ w_0 \neq 0, \ -\pi/2 < \alpha < \pi/2.$$

A domain D containing the origin is said to be α-spirallike if for each point $w_0 \neq 0$ in D the arc of the α-spiral from w_0 to the origin lies entirely in D. If $f(z)$ is analytic and univalent in Δ, with $f(0) = 0$, it is said to be α-spirallike if its range is α-spirallike.

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Let $\mathrm{Sp}(\alpha)$ denote the class of α-spirallike functions f with parameter α, $-\pi/2 < \alpha < \pi/2$. This class $\mathrm{Sp}(\alpha)$ is defined by the conditions

(1.1) $\quad f(0) = 0,\ f'(0) = 1$ and $\mathrm{Re}[e^{i\alpha} z f'(z)/f(z)] > 0$ for $|z| < 1$.

We note that $\mathrm{Sp}(0) = S^*$, the class of starlike functions with respect to the origin and $\mathrm{Sp}(\alpha) \subset S$, the class of normalized univalent regular functions in Δ.

In this note we observe the extremal and topological structure of the class $\mathrm{Sp}(\alpha)$ and we show that the class $\mathrm{Sp}(\alpha)$ is a compact subset of locally convex linear topological space A and give a detailed proof of the integral representation formula for the class $\mathrm{Sp}(\alpha)$. Also we obtain a result showing that a function in the class $\mathrm{Sp}(\alpha)$ can be approximated uniformly by functions of the form of finite Blaschke products on compact subsets of closed convex hull of $\mathrm{Sp}(\alpha)$. Finally we deal with an extremal problem for the class $\mathrm{Sp}(\alpha)$. This was commented by MacGregor [10] without proof and we give the proof in detail.

2. Extremal structure of the class $\mathrm{Sp}(\alpha)$

If $f \in \mathrm{Sp}(\alpha)$, the condition

$$\mathrm{Re}\left[e^{i\alpha}\frac{zf'(z)}{f(z)}\right] > 0, \quad |\alpha| < \pi/2, \tag{2.1}$$

has a nice geometric interpretation; With proper choice of arguments, the condition (2.1) is equivalent to

$$0 < \arg\left[ie^{i\alpha}\{re^{i\theta}f'(re^{i\theta})/f(re^{i\theta})\}\right] < \pi.$$

That is, the image under f of the circle $|z| = r$, $0 < r < 1$, is the curve C_r given by $w = f(re^{i\theta})$ and its radial angle is $\arg[izf'(z)/f(z)]$. So the condition (2.1) simply requires that this radial angle lies between α and $\alpha + \pi$. On the other hand, α-spirals are the curves with constant radial angle α, if oriented with increasing modulus.

An example is the function

$$\begin{aligned} f(z) &= \frac{z}{(1-z)^{2\tau}}, \quad \tau = \cos\alpha^{e^{-i\alpha}},\ |\alpha| < \pi/2 \\ &= \frac{z^{1-\tau}}{4^{\tau}}\left[\left(\frac{1+z}{1-z}\right)^2 - 1\right]^{\tau} \end{aligned}$$

which maps Δ onto the complement of an arc of an α-spiral. [6, p.55].

THEOREM 2.1. *The class of spirallike functions $Sp(\alpha)$ is a compact subset of locally convex topological space A.*

Proof. Since $\mathrm{Sp}(\alpha)$ is a subset of the compact subset S of A, it is enough to show that $\mathrm{Sp}(\alpha)$ is closed. Suppose that $f_n \in \mathrm{Sp}(\alpha)$ and that $f_n \to f$ uniformly on compact subsets of Δ. Then we have $f \in S$ and

$$\mathrm{Re}[e^{i\alpha} z f'(z)/f(z)] \geq 0, \quad z \in \Delta. \tag{2.2}$$

Now it remains only to show that the strict inequality holds in (2.2) for all $z \in \Delta$. If the function $e^{i\alpha} z f'(z)/f(z)$ is not constant, then, by the open mapping theorem, it is clear. If it is constant, then, since $f \in S$, we must have $f(z) = z$, and hence, from the condition that $|\alpha| < \pi/2$, we have

$$\mathrm{Re}[e^{i\alpha} z f'(z)/f(z)] = \cos\alpha > 0,$$

as desired.

LEMMA 1. *Let $\mathcal{P}$ be the class of functions p with positive real part and $p(0) = 1$ in Δ. $p \in \mathcal{P}$ if and only if there is a probability measure μ on $\partial\Delta$ such that*

$$p(z) = \int_{|x|=1} \frac{1 + xz}{1 - xz} d\mu(x) \quad (|z| < 1). \tag{2.3}$$

Proof. [7, p.30].

The following result for the class $\mathrm{Sp}(\alpha)$ is originally shown in [10] but we give another proof in detail.

THEOREM 2.2. *$f \in Sp(\alpha)$ if and only if there is a probability measure μ on $\partial\Delta$ such that*

$$f(z) = z \exp[\int_{|x|=1} -2\tau \log(1 - xz) d\mu(x)], \quad \tau = \cos\alpha e^{-i\alpha}. \tag{2.4}$$

The correspondence from the set of probability measures on $\partial\Delta$ to $Sp(\alpha)$ given by (2.4) is one-to-one.

Proof. Let $f \in \mathrm{Sp}(\alpha)$. The mapping from $\mathrm{Sp}(\alpha)$ to $\mathcal{P}$ defined by

$$p(z) = \sec\alpha[e^{i\alpha} z f'(z)/f(z) - i\sin\alpha], \quad |a| < \pi/2, \tag{2.5}$$

is a one-to-one correspondence from $\mathrm{Sp}(\alpha)$ onto $\mathcal{P}$. Since f is analytic in Δ, $f(0)=0$, $f'(0)=1$ and $p \in \mathcal{P}$ where p is given by (2.5) it follows that $f(z) \neq 0$ when $z \neq 0$. Thus $g(z)=\log[f(z)/z]$ is well defined and analytic in Δ with the choice of the branch so that $g(0)=0$. Since

$$\frac{d}{dz}[g(z)] = \cos\alpha e^{-i\alpha}\frac{p(z)-1}{z},$$

we conclude that

$$g(z) = \cos\alpha e^{-i\alpha}\int_0^z \frac{p(w)-1}{w}\,dw$$

and

$$f(z) = z\exp[\cos\alpha e^{-i\alpha}\int_0^z \frac{p(w)-1}{w}\,dw].$$

By using Lemma 1, we have

$$\begin{aligned} f(z) &= z\exp\left[\cos\alpha e^{-i\alpha}\int_0^z\int_{|x|=1}\left\{\frac{(1+xw)/(1-xw)-1}{w}\right\}d\mu(x)dw\right] \\ &= z\exp\left[\cos\alpha e^{-i\alpha}\int_{|x|=1}\left\{\int_0^z \frac{2x}{1-xw}dw\right\}d\mu(x)\right] \\ &= z\exp\left[\int_{|x|=1} -2\tau\log(1-xz)d\mu(x)\right] \text{ where } \tau=\cos\alpha e^{-i\alpha}. \end{aligned}$$

The one-to-one correspondence between $\mathrm{Sp}(\alpha)$ to $\{\mu\}$ on $\partial\Delta$ given by (2.4) may be viewed as equivalent to a uniqueness statement about moments. Suppose that p and q belong to $\mathcal{P}$ defined by (2.5) and correspond to the measures μ and ν given by (2.4), respectively. If we let

$$p(z) = 1+\sum_{n=1}^{\infty} p_n z^n \quad \text{and} \quad q(z) = 1+\sum_{n=1}^{\infty} q_n z^n$$

we find that $p \in \mathcal{P}$ if and only if $p(z)=\int_{|x|=1}\frac{1+xz}{1-xz}d\mu(x)$ implies that

$$p_n = 2\int_{|x|=1} x^n d\mu(x) \text{ and } q_n = 2\int_{|x|=1} x^n d\nu(x) \text{ for } n=1,2,\ldots.$$

So $p = q$ implies that if μ and ν are two probability measures on $\partial\Delta$ such that

$$\int_{|x|=1} x^n d\mu(x) = \int_{|x|=1} x^n d\nu(x) \quad \text{for} \quad n = 1, 2, \ldots,$$

then $\mu = \nu$ and conversely. Thus, the correspondence from the set of probability measures on $\partial\Delta$ to $\mathrm{Sp}(\alpha)$ given by (2.4) is one-to-one.

LEMMA 2 [7, P.47]. *Let Λ denote the set of probability measures on $\partial\Delta$. The set of extreme points of Λ consists of the point masses.*

LEMMA 3 [7, P.31]. *$p \in \mathcal{P}$ if and only if there exists a sequence of functions $\{p_n\}$ so that each p_n has the form*

$$q(z) = \sum_{k=1}^{m} t_k \frac{1 + x_k z}{1 - x_k z} \tag{2.6}$$

where $|x_k| = 1$, $t_k \geq 0$, $\sum_{k=1}^{m} t_k = 1$ and $p_n \to p$ uniformly on compact subsets of Δ.

Every function $f \in \mathrm{Sp}(\alpha)$ can be approximated uniformly on $|z| \leq r$ by functions of the form (2.7) as follows;

THEOREM 2.3. *$f \in Sp(\alpha)$ if and only if there is a sequence of functions $\{f_n\}$ having the form*

$$g(z) = \frac{z}{\Pi_{k=1}^{m}(1 - x_k z)^{2\tau t_k}} \tag{2.7}$$

where $|x_k| = 1$, $t_k \geq 0$, $\sum_{k=1}^{m} 2\tau t_k = 2\tau$, $\tau = \cos\alpha e^{-i\alpha}$, $|\alpha| < \pi/2$, and $f_n \to f$ uniformly on compact subsets of Δ.

Proof. According to Theorem 2.2, $f \in \mathrm{Sp}(\alpha)$ if and only if

$$\begin{aligned} f(z) &= z \exp\left[\tau \int_0^z \frac{p(w) - 1}{w} dw\right], \quad \tau = \cos\alpha e^{-i\alpha} \\ &= z \exp\left[\int_{|x|=1} -2\tau \log(1 - xz) d\mu(x)\right]. \end{aligned}$$

Since μ is a probability measure on $\partial\Delta$ there is a sequence of probability measures ν_n which are convex combinations of point masses so that

$$\int_{|x|=1} g(x)d\nu_n(x) \to \int_{|x|=1} g(x)d\mu(x)$$

for every continuous function g on $\partial\Delta$. (The proof of this fact depends on the Krein-Milman theorem, Lemma 2 and the fact that the set of probability measures on $\partial\Delta$ is compact in the weak star topology). We may write

$$\nu_n = \sum_{k=1}^{m} t_k \delta_{x_k} \quad \text{where} \quad t_k \geq 0,\ \sum_{k=1}^{m} t_k = 1$$

and δ_{x_k} denotes point mass at x_k (hence $|x_k| = 1$ for $k = 1, 2, \ldots, m$). If f_n denotes the functions in $\mathrm{Sp}(\alpha)$ corresponding to the measures ν_n it follows that $f_n \to f$ uniformly on compact subsets of Δ. The uniform convergence follows from Montel's theorem and the fact [p.733, 14] that

$$|f_n(z)| \leq \max_{|x|=1} \frac{|z|}{|1-xz|^{2\tau}},\ |f(z)| \leq \max \frac{|z|}{|1-xz|^{2\tau}},\ \tau = \cos\alpha e^{-i\alpha}.$$

Also, if $f_n \in \mathrm{Sp}(\alpha)$, we may write

$$f_n(z) = z \exp\left[\tau \int_0^z \frac{p_n(w)-1}{w} dw\right],\ p_n(w) \in \mathcal{P},\ \tau = \cos\alpha e^{-i\alpha}.$$

By applying Lemma 3 for $p_n \in \mathcal{P}$, we have

$$\begin{aligned} f_n(z) &= z \exp\left[\tau \int_0^z (\sum_{k=1}^{m} t_k \frac{1+x_k w}{1-x_k w} - 1)/w\, dw\right] \\ &= z \exp\left[\tau \int_0^z \{\sum_{k=1}^{m} t_k (\frac{1+x_k w}{1-x_k w} - 1)\}/w dw\right] \\ &= z \exp\left[2\tau \int_0^z \sum_{k=1}^{m} \frac{t_k x_k}{1-x_k w} dw\right] \end{aligned}$$

$$= z \exp[-2\tau \sum_{k=1}^{m} \log(1 - x_k z)]$$
$$= z \exp\left[\log \Pi_{k=1}^{m} (1 - x_k z)^{-2\tau t_k}\right]$$
$$= \frac{z}{\Pi_{k=1}^{m} (1 - x_k z)^{2\tau t_k}}$$

where $|x_k| = 1$, $t_k \geq 0$, $\sum_{k=1}^{m} 2\tau t_k = 2\tau$, $\tau = \cos\alpha e^{-i\alpha}$.

The converse follows from Theorem 2.1 and the fact that each g given by (2.7) is in $\mathrm{Sp}(\alpha)$.

3. Some remarks on Extreme points of $\overline{\mathrm{co}}\,\mathrm{Sp}(\alpha)$

Let $E\,\overline{\mathrm{co}}\,\mathrm{Sp}(\alpha)$ denote the set of extreme points of $\overline{\mathrm{co}}\,\mathrm{Sp}(\alpha)$. T. H. MacGregor [10] suggested a conjecture for the extreme points of the closed convex hulls of the class $\mathrm{Sp}(\alpha)$, $-\pi/2 < \alpha < \pi/2$, namely,

$$E\,\overline{\mathrm{co}}\,\mathrm{Sp}(\alpha) = \left\{ \frac{z}{(1 - xz)^{2\tau}} : |x| = 1,\ \tau = \cos\alpha e^{-i\alpha} \right\}. \tag{3.1}$$

In fact, if $f \in \mathrm{Sp}(\alpha)$, we have seen in Theorem 2.2 that

$$f(z) = z \exp\left[\int_{|x|=1} -2\tau \log(1 - xz) d\mu(x) \right],\ \tau = \cos\alpha e^{-i\alpha}. \tag{3.2}$$

The integral $\int_{|x|=1} -2\tau \log(1 - xz) d\mu(x)$ can be approximated locally uniformly in $\overline{\Delta}$ by sums $\sum_{k=1}^{m} -2\tau t_k \log(1 - x_k z)$ where $t_k \geq 0$, $\sum_{k=1}^{m} t_k = 1$ and $|x_k| = 1$. Consequently

$$\exp\left[\int_{|x|=1} -2\tau \log(1 - xz) d\mu(x) \right]$$

is approximated by the products

$$\prod_{k=1}^{m} (1 - x_k z)^{-2\tau t_k}, \quad \tau = \cos\alpha e^{-i\alpha}. \tag{3.3}$$

For complex number p with $\operatorname{Re} p > 0$, let $\mathcal{F}_p$ be the subclass of A of functions f given by

$$f(z) = \int_{|x|=1} \frac{1}{(1-xz)^p} d\mu(x)$$

for some probability measure μ on $\partial\Delta$. It is easily seen that each $\mathcal{F}_p$ class is closed and convex. For two complex numbers p, q with $\operatorname{Re} p > 0$, $\operatorname{Re} q > 0$ let the product $\mathcal{F}_p \cdot \mathcal{F}_q$ be given by

$$\mathcal{F}_p \cdot \mathcal{F}_q = \{fg : f \in \mathcal{F}_p,\ g \in \mathcal{F}_q\}.$$

The product of a function in $\mathcal{F}_p$ and a function in $\mathcal{F}_q$ can be written as an integral with respect to a probability measure on $\partial\Delta \times \partial\Delta$, the integrand being of the form $(1-xz)^{-p}(1-yz)^{-q}$, when $|x| = |y| = 1$. Suppose that for those two complex numbers p, q with $\operatorname{Re} p > 0$, $\operatorname{Re} q > 0$,

$$\mathcal{F}_p \cdot \mathcal{F}_q \subset \mathcal{F}_{p+q} \tag{3.4}$$

is true, then we obtain from (3.2) and (3.3) that

$$f(z) = \int_{|x|=1} \frac{z}{(1-xz)^{2\tau}} d\mu(x). \tag{3.5}$$

From (3.5) it is easy to verify that

$$\overline{\text{co}}\,\text{Sp}(\alpha) = \Big\{ \int_{|x|=1} \frac{z}{(1-xz)^{2\tau}} d\mu(x) : |x| = 1,\ \tau = \cos\alpha e^{-i\alpha} \Big\}$$

and

$$E\,\overline{\text{co}}\,\text{Sp}(\alpha) = \Big\{ \frac{z}{(1-xz)^{2\tau}} : |x| = 1,\ \tau = \cos\alpha e^{-i\alpha} \Big\}.$$

However, K. Pearce [12] showed that (3.4) is not the case in general. In his paper, Pearce showed that if

$$\mathcal{F}_p \cdot \mathcal{F}_q \subset \mathcal{F}_{p+q}$$

holds, then $p > 0,\ q > 0$ or else $p = q = 1 + it$ for some t real. Moreover, D. S. Moak [11] verified that if for $t \in \mathbb{R}$,

$$\mathcal{F}_{1+it} \cdot \mathcal{F}_{1+it} \subset \mathcal{F}_{2+2it}$$

then $t = 0$. This means that (3.4) is true only when $p > 0,\ q > 0$.

Therefore, for complex number p with $\operatorname{Re} p > 0$ the following statement does not hold:

$$\exp\left[\int_{|x|=1} -p\log(1 - xz)d\mu(x)\right] = \int_{|x|=1} (1 - xz)^{-p} d\nu(x).$$

But K. Pearce [12] verified that each function

$$f(z) = \frac{z}{(1 - xz)^{2\tau}},\ |x| = 1,\ \tau = \cos\alpha e^{-i\alpha},$$

uniquely maximizes the functional $\operatorname{Re} J_x$ over $\operatorname{Sp}(\alpha)$, where

$$J_x g = 2\overline{\tau}\,\overline{x} g''(0), \quad |x| = 1.$$

Hence, each function $f(z) = \frac{z}{(1-xz)^{2\tau}}$, $|x| = 1$, $\tau = \cos\alpha e^{-i\alpha}$, is necessarily an extreme point of $\overline{\text{co}}\operatorname{Sp}(\alpha)$, $-\pi/2 < \alpha < \pi/2$, that is,

$$\left\{\frac{z}{(1 - xz)^{2\tau}} : |x| = 1,\ \tau = \cos\alpha e^{-i\alpha}\right\} \subsetneq E\,\overline{\text{co}}\operatorname{Sp}(\alpha),\ 0 < |\alpha| < \pi/2.$$

4. Application to Extremal Problem

An application to the solution of extremal problem in the class of starlike functions S^* was given by G. M. Goluzin [17].

THEOREM (GOLUZIN). *For a given entire function $\Phi(w)$ and a given point z in $|z| < 1$ the maximum for the functional $Re[\Phi\{\log \frac{f(z)}{z}\}]$ in the class S^* is attained only for a function of the form*

$$f(z) = \frac{z}{(1 - e^{i\alpha}z)^2}, \quad \alpha \text{ real.}$$

We obtain a similar result for the class of spirallike functions. We begin by showing a subordination result for the class $Sp(\alpha)$ due to T. H. MacGregor [10], but here we give alternate proof for it. We need the following lemma due to R. M. Robinson [16].

LEMMA 4 (ROBINSON). *If $f(z)$ is subordinate to $g(z) = (1+z)/(1-z)$ in Δ, and if L is any continuous linear operator of order zero, then $L(f)$ is hull subordinate to $L(g)$ in Δ.*

THEOREM 4.1. *If $f \in Sp(\alpha)$, then $f(z)/z$ is subordinate to $F(z) = 1/(1-z)^{2\tau}$ where $\tau = \cos\alpha e^{-i\alpha}$.*

Proof. If $f \in \mathrm{Sp}(\alpha)$, we may write

$$e^{i\alpha}\frac{zf'(z)}{f(z)} = \cos\alpha p(z) + i\sin\alpha, \text{ where } p(z) \in \mathcal{P}.$$

Let $q(z) = zf'(z)/f(z) = e^{-i\alpha}[\cos\alpha p(z) + i\sin\alpha]$. Since $f(z) \neq 0$ when $z \in 0$, we have

$$\frac{d}{dz}[\log f(z)/z)] = \frac{q(z)-1}{z}$$

and

(4.1)

$$\log(f(z)/z) = \int_0^z \frac{q(w)-1}{w}dw \text{ where } q(w) = e^{-i\alpha}[\cos\alpha p(w) + i\sin\alpha].$$

The functions $\{q\}$ consists of those functions subordinate to

$$q_0(w) = e^{-i\alpha}[\cos\alpha\frac{1+w}{1-w} + i\sin\alpha]$$

which is the function mapping Δ onto a half plane. Also,

$$\begin{aligned}\int_0^z \frac{q_0(w)-1}{w}dw &= \int_0^z \frac{1}{w}[e^{-i\alpha}(\cos\alpha\frac{1+w}{1-w} + i\sin\alpha) - 1]dw \\ &= \int_0^z \frac{2\cos\alpha e^{-i\alpha}}{1-w}dw \\ &= -2\tau\log(1-z), \text{ where } \tau = \cos\alpha e^{-i\alpha}.\end{aligned}$$

The integral in (4.1) defines a continuous linear operator of order zero on the family $\{q\}$, and so we may apply Lemma 4. We can drop the hull in the subordination since

$$\int_0^z \frac{q_0(w)-1}{w}dw = -2\tau\log(1-z)$$

is univalent and convex in Δ. Therefore, we find that $f(z)/z$ is subordinate to $F(z) = 1/(1-z)^{2\tau}$, $\tau = \cos\alpha e^{-i\alpha}$.

THEOREM 4.2. *For a given non-constant entire function $\Phi(w)$ and a given point z in Δ, the maximum for the functional*

$$Re[\Phi\{\log f(z)/z\}]$$

over the class $Sp(\alpha)$ is attained only for a function of the form

$$f(z) = \frac{z}{(1 - xz)^{2\tau}}$$

where $|x| = 1$, $\tau = \cos\alpha e^{-i\alpha}$.

Proof. Since the class $\mathrm{Sp}(\alpha)$ is compact and the functional $\mathrm{Re}[\Phi\{\log f(z)/z\}]$ is continuous, this extremal problem has a solution in $\mathrm{Sp}(\alpha)$. Suppose that $f \in \mathrm{Sp}(\alpha)$. Theorem 4.1 implies that $f(z)/z$ is subordinate to $F(z) = 1/(1-z)^{2\tau}$, $\tau = \cos\alpha e^{-i\alpha}$, which is the same as $\log[f(z)/z]$ is subordinate to $\log F(z)$. Thus $g(z) = \Phi[\log f(z)/z]$ is subordinate to $G(z) = \Phi[-2\tau\log(1-z)]$, $\tau = \cos\alpha e^{-i\alpha}$. Also G is non-constant, since Φ is non-constant. If g is subordinate to the non-constant function G in Δ, then

$$g[\{z : |z| \le r\}] \subset D = G[\{z : |z| \le r\}]$$

for each r, $0 < r < 1$. The function $G(xz)$, $|x| \le 1$, is subordinate to $G(z)$ and so $\{g(z) : g \text{ is subordinate to } G \text{ in } \Delta\} = D$ where $r = |z|$ for any point in Δ. This uses the fact that $G(xz)$ is also of the form $f(z)/z$ where $f \in \mathrm{Sp}(\alpha)$. By considering a support line to the compact set D we conclude that

$$\max_{f\in\mathrm{Sp}(\alpha)} \mathrm{Re}\{\Phi[\log \frac{f(z)}{z}]\} = \mathrm{Re}\, w_1 \tag{4.2}$$

where $w_1 \in \partial\Delta$. Since G is an open mapping there is a number z_1 so that $|z_1| = r$ and $G(z_1) = w_1$. Of all the numbers w_1 satisfying (4.2) there is one for which $w_1 = \Phi[\log f(z)/z]$ where f is a given solution to the extremal problem. This is the precise situation described by Lemma 5 and so

$$\Phi[\log f(z)/z] = \Phi[-2\tau\log(1 - xz)], \ \tau = \cos\alpha e^{-i\alpha}. \tag{4.3}$$

That is, if f is a solution of the extremal problem, then (4.3) holds for some $x, |x| = 1$. Since Φ is non-constant we may write

$$\Phi(w) = c_0 + c_n w^n + c_{n+1} w^{n+1} + \cdots \quad \text{where } c_n \neq 0.$$

If we let $\log[f(z)/z] = \alpha_1 z + \alpha_2 z^2 + \cdots$ and $-2\tau \log(1 - xz) = \beta_1 z + \beta_2 z^2 + \cdots$ then (4.3) implies that $c_n \alpha_1^n = c_n \beta_1^n$. Since $c_n \neq 0$ this shows that $\alpha_1^n = \beta_1^n$ and so, in particular, $|\alpha_1| = |\beta_1|$. Since $\log[f(z)/z]$ is subordinate to $-2\tau \log(1 - xz)$, the equality $|\alpha_1| = |\beta_1|$ is only possible if

$$\log[f(z)/z] = -2\tau \log(1 - xyz)$$

for some y, $|y| = 1$. Therefore, we conclude that

$$f(z) = z/(1 - uz)^{2\tau}, \ |u| = 1, \ \tau = \cos\alpha e^{-i\alpha}$$

if f is a solution to the extremal problem. This completes the proof.

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