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THE STABILITY OF THE FREDHOLM INDEX IN A SEMI-PRIME RING

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In [1], Barnes introduced the generalized Fredholm theory in the context of a semi-prime ring: he gave a characterization of the Fredholm elements of a semi-prime ring, defined a generalized index on the semigroup of Fredholm elements and then proved the index product theorem for a semi-prime ring and the continuity of the index for a semi-prime Banach algebra. In this note we prove the stability of the index for a semi-prime ring.

Throughout this note, suppose A is a semi-prime ring (a ring which contains no nilpotent ideals) with unity $1 \neq 0$. If A has minimal ideals, the smallest ideal containing all of them is called the *socle* of A and is denoted by S_A. If A is a semi-prime ring then S_A always exists ([2, Proposition IV.31.10]).

For example, if $A = B(X)$ is the set of all bounded linear operators on a Banach space X, then A is a semi-prime ring with unity since every ideal contains finite rank operators and S_A is the set of finite rank operators.

We recall ([1], [3]) that if A is a semi-prime ring with unity 1, then an element $a \in A$ is called *Fredholm* if there are b_1 and b_2 in A for which

$$1 - b_1 a \in S_A$$

and

$$1 - ab_2 \in S_A.$$

Evidently, if $a, b \in A$, then

(0.1) $$a \text{ Fredholm}, \quad b \in S_A \Longrightarrow a + b \text{ Fredholm}.$$

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We also recall ([4], [5]) that $a \in A$ is called *regular* if there is a' in A for which

$$a = aa'a. \tag{0.2}$$

We write

$$A^\circ = \{a \in A \mid a \text{ is regular}\}.$$

If (0.2) holds we call a' a *generalized inverse* for a. If $A = B(X)$ and $a \in A$, then ([5, Theorem 6.4.4])

$$a \text{ Fredholm} \Longrightarrow a \text{ regular}. \tag{0.3}$$

However, if A is any semi-prime ring we cannot guarantee (0.3). Thus, to carry over much of the "spatial"Fredholm theory we need the condition (0.3). For this, we make the "regularity condition "

$$S_A \subseteq A^\circ. \tag{0.4}$$

It is clear that the regularity condition (0.4) is valid whenever $A = B(X)$. We assume throughout that $S_A \subseteq A^\circ$.

For brevity, we write

$$L_a : b \longmapsto ab \quad \text{from } A \text{ to } A$$

and

$$R_a : b \longmapsto ba \quad \text{from } A \text{ to } A$$

for the *left* and *right* multiplications associated with a.

Our first observation is elementary:

LEMMA 1. *If A is a semi-prime ring with unity 1 and $a \in A$, then a is Fredholm if and only if*

$$a \in A^\circ \text{ and } L_a^{-1}(0) \subseteq S_A \text{ and } R_a^{-1}(0) \subseteq S_A. \tag{1.1}$$

Proof. If $a = aa'a \in A$ satisfies (1.1), then

$$1 - a'a \in L_a^{-1}(0) \subseteq S_A \tag{1.2}$$

and

$$(1.3) \qquad 1 - aa' \in R_a^{-1}(0) \subseteq S_A,$$

which says that a is Fredholm. Conversely, suppose a is Fredholm and thus there is b in A for which

$$1 - ba \in S_A,$$

so that

$$a - aba \in S_A \subseteq A^\circ,$$

which, by [4, Theorem 1.4], gives $a \in A^\circ$. Assume $1 - ba = x \in S_A$. Then we have

$$c \in L_a^{-1}(0) \Longrightarrow ac = 0 \Longrightarrow bac = 0 \Longrightarrow (1-x)c = 0 \Longrightarrow c = xc \in xA,$$

which says that $L_a^{-1}(0) \subseteq S_A$ because xA is a right ideal in S_A. The argument for $R_a^{-1}(0) \subseteq S_A$ is similar.

We now consider the concept of the *order* for an ideal, a notion which plays the role of dimension in the ring theory.

A right (left) ideal J of A is of finite order if J is the sum of a finite number of minimal right (left) ideals of A. We recall ([1], [3]) that a right (left) ideal J is said to have order n if J can be written as the sum of n minimal right (left) ideals but of no fewer. For convenience, we define the order of zero ideal to be zero and write $\theta(J)$ for the order of an ideal J.

If $a \in A$ is Fredholm, then, by Lemma 1, both $L_a^{-1}(0)$ and $R_a^{-1}(0)$ have finite order since each ideal in S_A is the sum of a finite number of minimal ideals. We therefore define the *index* of a, index(a), as follows ([1], [3]) :

$$(1.4) \qquad \operatorname{index}(a) = \theta(L_a^{-1}(0)) - \theta(R_a^{-1}(0)).$$

For example, if $A = B(X)$ and $a \in A$ is Fredholm then

$$\theta(L_a^{-1}(0)) = \dim a^{-1}(0)$$

and

$$\theta(R_a^{-1}(0)) = \dim X/a(X);$$

therefore in the case the index given by (1.4) reduces to the usual index. Barnes showed ([1, Theorem 3.2]) that if a and b are Fredholm elements in A, then so is ab and index(ab) = index(a)+index(b) (the index product theorem) and also ([1, Theorem 4.1]) that if A is a semi-prime Banach algebra, then the index is a continuous function on the open semi-group of Fredholm elements.

We now show that the index is stable under the "socle"perturbation. We begin with:

DEFINITION 2. We shall call $a \in A$ a Weyl element if

$$a \text{ is Fredholm and index}\,(a) = 0.$$

It is well known ([5, Theorem 6.5.2]) that if $T \in B(X)$, then

$$T = S + P \quad \text{with invertible } S \text{ and finite rank } P \Longrightarrow T \text{ Weyl.} \tag{2.1}$$

We have an analogue of (2.1) for a semi-prime ring:

THEOREM 3. *If A is a semi-prime ring with unity 1 and $a \in A$, then*

$$a \in A^{-1} + S_A \Longrightarrow a \textit{ Weyl}, \tag{3.1}$$

where A^{-1} denotes the set of all invertible elements of A.

Proof. Suppose $a \in A^{-1} + S_A$ and thus we can write

$$a = b(1+s) \quad \text{with } b \in A^{-1} \text{ and } s \in S_A.$$

By (0.1), $1+s$ is Fredholm, and hence the index product theorem gives

$$\text{index}\,(a) = \text{index}\,(b) + \text{index}\,(1+s) = \text{index}\,(1+s).$$

Thus it is sufficient to show that $1+s$ is Weyl. We now claim that both L_{1+s} and R_{1+s} are of finite ascent. For if not, then there is an orthogonal sequence of minimal independent idempodents (x_n) in S_A for which, for each $n \in \mathbf{N}$,

$$x_n \in (L_{1+s})^{-n-1}(0) \text{ and } x_n \notin (L_{1+s})^{-n}(0).$$

Thus $(L_{1+s})^{n+1}(x_n) = 0$, and hence $(1+s)^{n+1}(x_n) = 0$, so that $x_n \in sA$. This contradicts the fact that sA is of finite order. Therefore L_{1+s} is of finite ascent. A similar argument gives R_{1+s} is of finite ascent. If L_{1+s} is of finite ascent p and R_{1+s} is of finite ascent q, then the index product theorem gives

$$\begin{aligned} (3.2) \qquad n \operatorname{index}(1+s) &= \operatorname{index}((1+s)^n) \\ &= \theta(L^{-1}_{(1+s)^n}(0)) - \theta(R^{-1}_{(1+s)^n}(0)) \\ &= \theta((L_{1+s})^{-n}(0)) - \theta((R_{1+s})^{-n}(0)) \\ &= \theta((L_{1+s})^{-p}(0)) - \theta((R_{1+s})^{-q}(0)), \end{aligned}$$

where $n \geq \max(p, q)$. Since (3.2) is independent of $n \geq \max(p, q)$ it follows that $\operatorname{index}(1 + s) = 0$, which says that $1 + s$ is Weyl. This completes the proof.

We are ready for the stability of the index:

THEOREM 4. *If A is a semi-prime ring with unity 1, then*

$$(4.1) \qquad a \text{ Fredholm}, \ b \in S_A \Longrightarrow \operatorname{index}(a+b) = \operatorname{index}(a).$$

Proof. Suppose $a \in A$ is Fredholm with generalized inverse $a' \in A$. It thus follows from (1.2) and (1.3) that

$$1 - a'a \in S_A \quad \text{and} \quad 1 - aa' \in S_A,$$

which says that a' is also Fredholm. Thus the index product theorem gives

$$\operatorname{index}(a) = \operatorname{index}(a) + \operatorname{index}(a') + \operatorname{index}(a),$$

and hence

$$\operatorname{index}(a') = -\operatorname{index}(a).$$

We now observe that if $b \in S_A$, then Theorem 3 gives

$$a'(a+b) = (1+a'b) - (1-a'a) \text{ is Weyl}$$

because $a'b - (1-a'a) \in S_A$. Thus $\operatorname{index}(a'(a+b)) = 0$, so that

$$\operatorname{index}(a+b) = -\operatorname{index}(a') = \operatorname{index}(a).$$

We conclude with:

COROLLARY 5. *Let A be a semi-prime ring with unity 1. If $a \in A$ is Fredholm with generalized inverse $a' \in A$ and $b \in A$, then*

(5.1) $1 + a'b$ Fredholm $\Longrightarrow \operatorname{index}(a+b) = \operatorname{index}(a) + \operatorname{index}(1+a'b)$.

Proof. If $1 + a'b$ is Fredholm, then, by (0.1),

$$a'(a+b) = 1 + a'b - (1-a'a) \text{ is Fredholm.}$$

Thus Theorem 4 gives

$$\operatorname{index}(a'(a+b)) = \operatorname{index}(1 + a'b - (1-a'a)) = \operatorname{index}(1+a'b),$$

so that

$$\operatorname{index}(a+b) = \operatorname{index}(1+a'b) - \operatorname{index}(a') = \operatorname{index}(a) + \operatorname{index}(1+a'b).$$

The proof of the continuity of the index ([1, Theorem 4.1]) is a rather long technical argument in manipulating idempotents and ideals. However our corollary 5 gives a simple proof with the assumption that $S_A \subseteq A^{\circ}$. Because, if A is a semi-prime Banach algebra and if $b \in A$ has sufficiently small norm, then $1 + a'b$ is invertible, so that (5.1) says that

$$\operatorname{index}(a+b) = \operatorname{index}(a),$$

which says that the index is continuous function in the open semigroup of Fredholm elements of A.

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