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MINIMAL FREE RESOLUTIONS OF PERFECT AND ALMOST COMPLETE INTERSECTION GRADED IDEALS

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1. Introduction

Let $R = \oplus_{i=0}^{\infty} R_i$ be a polynomial ring in n indeterminates over a field $k : R = k[X_1, \dots, X_n]$. We consider the usual grading on R determined by the total degree of polynomials. Then each graded piece R_i of R is a k-vector space of dimension $\binom{i+n-1}{n-1}$ for all $i \geq 0$ and R_1 generates R as a k-algebra. The ideal $\underline{m} = \oplus_{i>0} R_i$ is the unique maximal homogeneous ideal of R, and R can be treated as if it were an ordinary local ring (See [3]).

By analogy with the local case, if

$$\cdots \to F_i \xrightarrow{d_i} F_{i-1} \to \cdots \to F_1 \xrightarrow{d_1} F_0 \xrightarrow{\varepsilon} M \to 0 \tag{1}$$

is a free resolution of an R-module M, and we denote

$$N_i = \operatorname{Im} d_i = \begin{cases} \operatorname{Ker} d_{i-1} & \text{for } \ i \geq 2 \\ \operatorname{Ker} \varepsilon & \text{for } \ i = 1, \end{cases}$$

then we call (1) a minimal free resolution if $N_i \subseteq \underline{m} F_{i-1}$ for all $i \geq 1$ (See Ch.7 of [5]).

By graded R-module we mean an R-module M with a decomposition by k-vector spaces, $M = \oplus_{i \in \mathbb{Z}} M_i$ (with $M_i = 0$ for all $i < \alpha$ and some $\alpha \in \mathbb{Z}$), compatible with the R-module structure, which means $R_i M_j \subseteq M_{i+j}$ for all i and j. A graded homomorphism is a homogeneous homomorphism of graded R-modules of degree 0. A graded (minimal) free resolution of a graded R-module M is a resolution like (1), with

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all F_i's graded R-modules and all d_i's (and ε) graded homomorphisms. If M is finitely generated, then every F_i has to be of the form $F_i = \oplus_{j=1}^{r}(R(-\nu_{i,j}))^{\alpha_{i,j}}$, where the $\nu_{i,j}$'s and the $\alpha_{i,j}$'s are, respectively, the degrees of the generators of $N_i = \operatorname{Im} d_i$ and the number of generators in each degree. We call the numbers $\nu_{i,j}$ the twisting numbers of M and each $\alpha_{i,j}$ the multiplicity of $\nu_{i,j}$ (at F_i). The numbers,

$$b_i = \sum_j \alpha_{i,j} = \operatorname{rank} F_i = \dim_k \operatorname{Tor}_i^R(M,k)$$

(=minimal number of generators of N_i) are called the Betti numbers of M.

In [1], Eisenbud and Goto calculated Betti numbers of R/I, when I is a graded ideal of R and R/I has p-linear resolution. Lorenzini [4] had partial results about twisting numbers and multiplicities and Betti numbers of a Cohen-Macaulay ring R/I.

Based on the work of Lorenzini, in this paper we get some results about Betti numbers when I is a perfect and almost complete intersection ideal.

Finally, for the following sections we define the Hilbert function of M. Let $M = \oplus_{i\in\mathbb{Z}} M_i$ be a finitely generated graded R-module. The Hilbert function of M is the function $H(M,\cdot) : \mathbb{Z} \to \mathbb{Z}$ defined by $H(M,i) = \dim_k M_i$ ($\forall i \in \mathbb{Z}$). We also define the first difference of the Hilbert function of M as

$$\Delta H(M,i) = H(M,i) - H(M,i-1), \quad \forall i \in \mathbb{Z}.$$

Inductively, for every $r > 1$, define the r-th difference of the Hilbert function of M as

$$\Delta^r H(M,i) = \Delta^{r-1} H(M,i) - \Delta^{r-1} H(M,i-1), \quad \forall i \in \mathbb{Z}.$$

2. Basic Theorems

The result of this section is mostly due to [4], and we deal with a perfect graded ideal I of $R = k[X_1,\ldots,X_n]$. When the length of a maximal regular sequence in I is equal to the homological dimension of

R/I, I is defined as a perfect ideal. Note that I is perfect in R if and only if R/I is a Cohen-Macaulay ring. Suppose I has height s. Then the homological dimension of I is $s-1$. Hence I has graded minimal free R-resolution:

$$(2) \qquad 0 \to F_{s-1} \to \cdots \to F_i \xrightarrow{d_i} F_{i-1} \to \cdots \to F_0 \to I \to 0$$

Since $F_i \otimes_R R/\underline{m} = \mathrm{Tor}_i^R(I,k)$ $(\forall i \geq 1)$, the information on $\mathrm{Tor}_i^R(I,k)$ gives the twisting numbers of I.

Define

$$\alpha(I) = \min\{t | I_t \neq (0)\}.$$

Assume k is infinite, and we may assume that $X_{s+1}, \ldots, X_n$ is a regular sequence modulo I. Put $A = R/I$ and

$$B = \frac{A}{(X_{s+1}, \ldots, X_n)A} \cong \frac{R}{(I, X_{s+1}, \ldots, X_n)}.$$

Define

$$\begin{aligned}\sigma(I) &= \min\{t | \Delta^{n-s} H(A,t) = 0\} \\ &= \min\{t | B_t = (0)\}.\end{aligned}$$

It is clear that if $G_1, \ldots, G_h$ is a minimal set of generators of I, then

$$\min\{\deg G_i \,|\, i = 1, \ldots, h\} = \alpha(I)$$

and it can easily be proved that $\max\{\deg G_i \,|\, i = 1, \ldots, h\} \leq \sigma(I)$.

THEOREM 2.1. *Let I be a height s, perfect graded ideal of $R = k[X_1, \ldots, X_n]$ and suppose $\alpha(I) = d$, $\sigma(I) = d + r$. Then $\mathrm{Tor}_i^R(I,k)$ vanishes in every degree different from $d+i, d+i+1, \ldots, d+i+r$.*

Proof. See Theorem 2.2 of [4].

From Theorem 2.1, we can deduce that in the minimal free resolution of I, (2) over R,

$$\begin{aligned}(3) \quad F_i = R(-(d+i))^{\alpha_{i,0}} \oplus R(-(d+i+1))^{\alpha_{i,1}} \oplus \cdots \\ \cdots \oplus R(-(d+i+r-1))^{\alpha_{i,r-1}} \oplus R(-(d+i+r))^{\alpha_{i,r}}\end{aligned}$$

for each $i = 0, \ldots, s-1$, with the $\alpha_{i,j}$'s not necessarily all different from 0. If we put $N_i = \operatorname{Im} d_i$, for every $i \geq 1$, and $N_0 = I$, then we have that $\alpha_{i,0} = H(N_i, d+i)$ and that N_i is generated at most in degrees $d+i, \ldots, d+i+r = \sigma(I)+i$, for all $i = 0, \ldots, s-1$.

Now, dualize resolution (2), by applying the functor $(\cdot)^* = \operatorname{Hom}(\cdot, R)$, and we obtain

(2)*

$$0 \to R \xrightarrow{\partial_0} F_0^* \to \cdots F_{i-1}^* \xrightarrow{\partial_i} F_i^* \to \cdots \to F_{s-1}^* \to \operatorname{Ext}_R^s(A, R) \to 0$$

which is a graded minimal free resolution of $\operatorname{Ext}_R^s(A, R)$ and

$$F_i^* = \oplus_{j=1}^r R(\nu_{i,j})^{\alpha_{i,j}}, \quad \forall i = 0, \ldots, s-1.$$

By (3),

$$F_i^* = R(d+i+r)^{\alpha_{i,r}} \oplus R(d+i+r-1)^{\alpha_{i,r-1}} \oplus \cdots$$
$$\cdots \oplus R(d+i+1)^{\alpha_{i,1}} \oplus R(d+i)^{\alpha_{i,0}}.$$

Put $L_i = \operatorname{Im} \partial_i$, for each $i = 1, \ldots, s-1$, and $L_s = \operatorname{Ext}_R^s(A, R)$. Now, for any $i = 0, \ldots, s-1$, and every $t = 1, \ldots, r$, let $W_t(N_i)$ denote the vector subspace of $(N_i)_{d+i+t}$ generated by $(N_i)_{d+i+t-1}$ under multiplication by $X_1, \ldots, X_n$, i.e.,

$$W_t(N_i) = X_1(N_i)_{d+i+t-1} + \cdots + X_n(N_i)_{d+i+t-1} \subseteq (N_i)_{d+i+t}.$$

It is clear that for each $i \geq 0$, and $t = 1, \ldots, r$,

$$\dim_k W_t(N_i) = H(N_i, d+i+t) - \alpha_{i,t}.$$

Similarly, consider the dual resolution and for any $i = 1, \ldots, s$ and every $t = 1, \ldots, r$, define $W_t(L_i)$ as the k-vector subspace of $(L_i)_{-(d+i-1+r-t)}$ generated by $(L_i)_{-(d+i+r-t)}$ under multiplication by $X_1, \ldots, X_n$, i.e.,

$$W_t(L_i) = X_1(L_i)_{-(d+i+r-t)} + \cdots + X_n(L_i)_{-(d+i+r-t)}$$
$$\subseteq (L_i)_{-(d+i-1+r-t)}.$$

Again, it is clear that

$$\dim_k W_t(L_i) = H(L_i, -(d+i-1+r-t)) - \alpha_{i-1,r-t}.$$

THEOREM 2.2. *Let I be a perfect homogeneous ideal of R of height s, with $\alpha(I) = d$, $\sigma(I) = d+r$; and let $\alpha_{i,j}$ be the multiplicity of $d+i+j$ at F_i ($j = 0, \dots, r; i = 0, \dots, s-1$). Then;*

(a) *for any $i = 1, \dots, s-1$ and any $t = 0, \dots, r-1$, $\alpha_{k,0} = \alpha_{k,1} = \cdots = \alpha_{k,t} = 0$, $\forall k = i, \dots, s-1$, if and only if*

$$\dim_k W_{t+1}(N_{i-1}) = \sum_{j=0}^{t} \binom{t+1-j+n-1}{n-1} \alpha_{i-1,j};$$

(b) *for any $i = 1, \dots, s-1$, and any $t = 0, \dots, r-1$, $\alpha_{k,r} = \alpha_{k,r-1} = \cdots = \alpha_{k,r-t} = 0$, $\forall k = 0, \dots, i-1$, if and only if*

$$\dim_k W_{t+1}(L_{i+1}) = \sum_{j=0}^{t} \binom{t+1-j+n-1}{n-1} \alpha_{i,r-j}.$$

Proof. See Theorem 3.2 of [4].

3. Applications

In this section, let I be a perfect graded ideal of $R = k[X_1, \dots, X_n]$ with $ht(I) = s$, and an almost complete intersection. I is defined as an almost complete intersection ideal when the number of minimal generators of I is $ht(I) + 1$. Then it is known that $I = (G_1, \dots, G_s, K)$, where homogeneous elements $G_1, \dots, G_s$ is a regular sequence and a homogeneous element $K \notin (G_1, \dots, G_s)$.

(*) Assume $\deg G_i = d$, for any $i = 1, \dots, s$ and $d < \deg K = d+r \leq 2d-1$, with $d > 1$, $r > 1$.

THEOREM 3.1. *Let I be a perfect, almost complete intersection with the condition (*). Then multiplicities $\alpha_{i,j}$'s of I in (3) are following;*

$$\alpha_{i,0} = \alpha_{i,1} = \cdots = \alpha_{i,r-1} = 0, \quad \forall i = 1, \dots, s-1.$$

Proof. Let $J = (G_1, \dots, G_s)$. Since $\alpha_{0,1} = \alpha_{0,2} = \cdots = \alpha_{0,r-1} = 0$, we get $W_t(N_0) = J_{d+t}$ for any $t = 1, \dots, r$. From the fact that $G_1, \dots, G_s$ is a regular sequence, and $r \leq d-1$

$$\dim_k W_t(N_0) = s\binom{t+n-1}{n-1}, \quad \forall t = 1, \dots, r.$$

Now $\dim_k W_t(N_0) = \sum_{j=0}^{t-1} \binom{t-j+n-1}{n-1}\alpha_{i-1,j}$, since $\alpha_{0,1} = \cdots = \alpha_{0,r-1} = 0$, and $\alpha_{0,0} = s$. Hence, by Theorem 2.2(a), $\alpha_{i,0} = \alpha_{i,1} = \cdots = \alpha_{i,r-1} = 0$, $\forall i = 1, \ldots, s-1$.

COROLLARY 3.2. *With the same I as in Theorem 3.1, $F_0 = R(-d)^s \oplus R(-d-r)$, $F_i = R(-d-i-r)^{\alpha_{i,r}}$, $i = 1, \ldots, s-1$. Hence the Betti numbers $b_0 = s+1$, and $b_i = \alpha_{i,r}$, and the minimal resolution (2) is linear except at F_0. (For the linearity of the minimal resolution, refer [1]).*

Next, consider a duality of Theorem 3.1 in the minimal free resolution $(2)^*$ of $\mathrm{Ext}_R^s(A, R) = L_s$. By (Proposition 5, Ch.IV of [6]),

$$L_s \cong \frac{(G_1, \ldots, G_s) : I}{(G_1, \ldots, G_s)}$$

and L_s is generated at most in degrees $-(d+s-1+r), \ldots, -(d+s-1)$.

Since $G_1, \ldots, G_s$ is a regular sequence, $S = R/(G_1, \ldots, G_s)$ is Gorenstein. Now $I/(G_1, \ldots, G_s) = (G_1, \ldots, G_s, K)/(G_1, \ldots, G_s) = (\overline{K}) \neq 0$ in S and $\dim S = \dim S/(\overline{K})$, hence by Proposition 3.1 of [2], we get $\mathrm{Ann}_S(\mathrm{Ann}_S(\overline{K})) = (\overline{K})$. Therefore $\mathrm{Ann}_R(L_s) = I$.

THEOREM 3.3. *Let I be same as in Theorem 3.1. Further Assume $L_s = (\alpha) \oplus M$, where $\deg \alpha = -(d+s-1+r)$ with $\mathrm{Ann}_R(\alpha) = I$ and $M \subset \oplus_{i \geq -(d+s-1)}(L_s)_i$. Then*

$$\alpha_{i,r} = \alpha_{i,r-1} = \cdots = \alpha_{i,1} = 0, \quad \textit{for any} \quad i = 0, \ldots, s-2.$$

Proof. $W_{t+1}(L_s) = R_{t+1}\alpha$, for any $t = 0, \ldots, r-1$, and multiplication by α gives an injection since $\mathrm{Ann}_R(\alpha) = I$, and $r \leq d-1$. Hence $W_{t+1}(L_s) \cong R_{t+1}$, and

$$\begin{aligned}\dim_k W_{t+1}(L_s) &= \binom{t+1+n-1}{n-1} \\ &= \sum_{j=0}^{t} \binom{t+1-j+n-1}{n-1}\alpha_{s-1}, r-j, \ \forall t = 0, \ldots, r-1\end{aligned}$$

since $\alpha_{s-1,r} = 1, \alpha_{s-1,r-1} = \cdots = \alpha_{s-1,1} = 0$. Therefore by Theorem 2.2(b), $\alpha_{i,r} = \alpha_{i,r-1} = \cdots = \alpha_{i,1} = 0$, $i = 0, \ldots, s-2$.

COROLLARY 3.4. *With the same I as in Theorem 3.3, $F_i = R(i+d)^{\alpha_{i,0}}$, $i = 0, \ldots, s-2$, and $F_{s-1} = R(s-1+d+r) \oplus R(s-1+d)^{\alpha_{s-1,0}}$. Therefore the minimal resolution (2) is linear except at F_{s-1}.*

References

1. D. Eisenbud and S. Goto, *Linear free resolutions and minimal multiplicity*; J. Algebra **88** (1984), 89–133.
2. E. Kunz, *Almost Complete Intersections Are Not Gorenstein Rings*; J. Algebra **28** (1974), 111–115.
3. A. Lorenzini, *Graded rings*, Queen's Papers in Pure and Appl. Math. vol. 61, 1982.
4. ______, *Betti numbers of perfect homogeneous Ideals*; J. Pure and Appl. Algebra **60** (1989), 273–288.
5. H. Matsumura, *Commutative ring theory*, Cambridge Univ. Press, Cambridge 1986.
6. J. Serre, *Algèbre local-Mutiplicités*, Lecture Notes in Math., vol. 11, Springer-Verlag, Berlin 1965.

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