

ON THE AUTOMORPHISM GROUPS OF PSEUDO- f -MANIFOLDS

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0. Introduction

Given a geometric structure on a differentiable manifold, the transformation group leaving the structure invariant is often a Lie transformation group and the problem has been studied by many authors [2, 4, 5, 6, 7, 9, 11]. Chu and Kobayashi [6] have summarized these results in chronological order and given systematic proofs.

Recently, one of the present authors [1] and Matsumoto [8] have introduced an pseudo- f -structure defined by f satisfying the equation $f^3 - f = 0$. A pseudo- f -structure is a generalization of an almost product structure and an almost paracontact structure.

The purpose of this paper is to investigate the automorphism groups of a compact almost paracontact manifold and a compact pseudo- f -manifold.

Making use of the theorems due to Bochner [3] and Palais [10], we prove the following results:

THEOREM A. *The automorphism group of a compact almost paracontact metric manifold is a Lie transformation group.*

THEOREM B. *The automorphism group of a compact pseudo- (f, g) -manifold is a Lie transformation group.*

THEOREM C. *The automorphism group of a compact almost product metric manifold is a Lie transformation group.*

In these cases, the topology of the Lie group is one of uniform convergence of functions together with their partial derivatives through the third order and it is stronger than the compact open topology.

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1. Automorphisms of an almost paracontact manifold

An almost paracontact structure is defined by three tensor fields; a (1, 1)-tensor field f , a vector field ξ and 1-form η . They satisfy the following conditions:

$$(1.1) \quad f^2X = X - \eta(X)\xi,$$

$$(1.2) \quad f\xi = 0, \quad \eta(fX) = 0, \quad \eta(\xi) = 1$$

for any vector field X on a manifold M . A manifold M with such a structure is called an almost paracontact manifold [8, 12].

If M has an associated Riemannian metric g such that

$$(1.3) \quad g(fX, fY) = g(X, Y) - \eta(X)\eta(Y),$$

$$(1.4) \quad g(X, \xi) = \eta(X)$$

for any vector fields X and Y on M , then M is called an almost paracontact metric manifold. If we put

$$(1.5) \quad F(X, Y) = g(fX, Y)$$

for any vector fields X and Y on M , then F is a symmetric covariant tensor field. Let F_{ji} be the components of F with respect to the local coordinate system $\{U, x^k\}$.

We now define an automorphism of an almost paracontact metric manifold M . A diffeomorphism h of M onto itself is called an automorphism of M if the diffeomorphism h leaves the tensor fields F and η invariant, that is,

$$(1.6) \quad h^*F = F, \quad h^*\eta = \eta,$$

where h^* denotes the codifferential of h . It is easily seen that the set of all automorphisms of M forms a group of transformations on M . We denote the group by $G(F, \eta)$.

2. Proof of Theorem A.

First of all, we now state well known theorems for later use.

THEOREM of Bochner [3]. *Let S be a space of vector fields X on a compact manifold M such that, for every point of M , there is a system of elliptic partial differential equations*

$$g^{ji}\nabla_j\nabla_iX^r + h_i{}^{rk}\nabla_kX^i + h_i{}^rX^i = 0, \quad i=1, \dots, n,$$

defined in a neighborhood of that point and satisfied by all X . Then the dimension of S is finite.

THEOREM of Palais [10]. *Let G be a group of diffeomorphisms of a manifold M . Let S be the set of all vector fields X on M which generate global 1-parameter groups $\phi_t = \exp tX$ of transformations of M such that $\phi_t \in G$. If S generates a finite-dimensional Lie algebra, then G is a Lie transformation group and S is the Lie algebra of G .*

Making use of these theorems, we shall prove Theorem A. Let $A(F, \eta)$ be the set of all infinitesimal automorphism X of F and η on M . A vector field X is an infinitesimal automorphism of F and η if and only if

$$(2.1) \quad L_X F = 0, \quad L_X \eta = 0,$$

where L_X denotes the Lie derivative with respect to X , The set $A(F, \eta)$ is a Lie subalgebra of $\mathfrak{X}(M)$, the Lie algebra of all vector fields on M . Since M is compact, any infinitesimal automorphism X in $A(F, \eta)$ is complete. Hence X generates a global 1-parameter group of transformations ϕ_t of M . Moreover, it follows from the definition of the Lie derivative that ϕ_t is an automorphism in $G(F, \eta)$. By Theorem of Palais, $G(F, \eta)$ is a Lie transformation group if $A(F, \eta)$ is finite-dimensional. Therefore, we shall prove that $A(F, \eta)$ is finite-dimensional.

For any infinitesimal automorphism X in $A(F, \eta)$, from the first equation of (2.1) we get

$$(2.2) \quad L_X \nabla_j F_{ik} = - (L_X \{^h_{ij}\}) F_{hk} - (L_X \{^h_{ij}\}) F_{ih},$$

where $\{^h_{ij}\}$ denotes the Christoffel symbol and ∇_j the covariant derivative with respect to the Riemannian connection. The left side of (2.2) becomes

$$X^h \nabla_h \nabla_j F_{ik} + \nabla_h F_{ik} \cdot \nabla_j X^h + \nabla_j F_{hk} \cdot \nabla_i X^h + \nabla_j F_{ih} \cdot \nabla_k X^h,$$

which does not involve second derivatives of X . Hence (2.2) is written by

$$(2.3) \quad (F_{hk} \nabla_j \nabla_i + F_{ih} \nabla_j \nabla_k) X^h + \dots = 0,$$

where the dots \dots indicate terms not involving second derivatives of X . Transvecting (2.3) with g^{ik} , we get

$$(2.4) \quad F_h^i \nabla_j \nabla_i X^h + \dots = 0,$$

where $F_h^i = g^{ik} F_{kh}$.

Transvecting (2.3) with g^{ji} , we get

$$(2.5) \quad F_{hk} g^{ji} \nabla_j \nabla_i X^h + F_h^j \nabla_j \nabla_k X^h + \dots = 0.$$

Taking account of (2.4), we have

$$F_{hk} g^{ji} \nabla_j \nabla_i X^h + F_h^j (\nabla_j \nabla_k X^h - \nabla_k \nabla_j X^h) + \dots = 0,$$

from which

$$F_{hk} g^{ji} \nabla_j \nabla_i X^h + F_h^j R_{jkl} X^l + \dots = 0.$$

Operating F^{rk} to the last equation, we obtain

$$(2.6) \quad g^{ji} \nabla_j \nabla_i X^r - \xi^r \eta_h g^{ji} \nabla_j \nabla_i X^h + \dots = 0.$$

On the other hand, from the second equation of (2.1) we get

$$L_X \nabla_j \eta_i = -(L_X \{_{ji}^h\}) \eta_h = -\eta_h \nabla_j \nabla_i X^h - \eta_h R_{kji} X^k,$$

from which

$$(2.7) \quad \eta_h \nabla_j \nabla_i X^h + \dots = 0.$$

Substituting (2.7) into the equation (2.6), we get

$$(2.8) \quad g^{ji} \nabla_j \nabla_i X^r + \dots = 0.$$

Thus, we have a system of elliptic differential equations satisfied by all infinitesimal automorphisms X in $A(F, \eta)$. By Theorem of Bochner, $A(F, \eta)$ is finite-dimensional. This completes the proof of Theorem A.

3. Automorphisms of a pseudo- f -manifold

A pseudo- f -structure on a differentiable manifold M is defined by a tensor field f of type (1, 1) satisfying the equation:

$$(3.1) \quad f^3 - f = 0.$$

If the rank of f is constant everywhere and is equal to r , then we call the structure a pseudo- f -structure of rank r and M is called a pseudo- f -manifold of rank r [1, 12]. A pseudo- f -structure is a generalization of an almost product structure ($r=n$) and an almost paracontact structure ($r=n-1$).

If we put

$$(3.2) \quad s = f^2, \quad t = -f^2 + 1,$$

then the operators s and t acting to the tangent space at each point of M are complementary projections and there exist complementary distributions S and T corresponding to s and t , respectively. If the rank of f is r , then S is r -dimensional and T is $(n-r)$ -dimensional.

If there exists an associated Riemannian metric g such that

$$(3.3) \quad g(fX, fY) = g(X, Y) - g(tX, tY)$$

for any vector fields X and Y on M , then M is called a pseudo- (f, g) -manifold. In a pseudo- (f, g) -manifold, we put

$$(3.4) \quad F(X, Y) = g(fX, Y),$$

$$(3.5) \quad w(X, Y) = g(tX, Y)$$

for any vector fields X and Y on M , then F and w are symmetric covariant tensor fields.

We now define an automorphism of a pseudo- (f, g) -manifold M . A diffeomorphism h of M onto itself is called an automorphism of M if the diffeomorphism h leaves the covariant tensor fields F and w invariant, that is,

$$(3.6) \quad h^*F = F, \quad h^*w = w.$$

It is easily seen that the set of all automorphisms of M forms a group of transformations on M . we denote the group by $G(F, w)$.

4. Proof of Theorem B.

Making use of the theorems due to Bochner and Palais, we shall prove Theorem B. Let $A(F, w)$ be the set of all infinitesimal automorphism X of F and w on a pseudo- f -manifold M . A vector field X is an infinitesimal automorphism of F and w if and only if

$$(4.1) \quad L_X F = 0, \quad L_X w = 0$$

The set $A(F, w)$ is a Lie subalgebra of $\mathfrak{X}(M)$, the Lie algebra of all vector fields on M . By Theorem of Palais, $G(F, w)$ is a Lie transformation group if $A(F, w)$ is finite dimensional. Therefore, we proceed to show that $A(F, w)$ is finite dimensional.

For any infinitesimal automorphism X in $A(F, w)$, from the first equation of (4.1) we get

$$(4.2) \quad F_{hk}g^{ji}\nabla_j\nabla_i X^h + \dots = 0$$

by calculation similar to that of section 2. Operating F^{rk} to (4.2) we obtain

$$(4.3) \quad g^{ji} \nabla_j \nabla_i X^r - w_k^r g^{ji} \nabla_j \nabla_i X^h + \dots = 0,$$

where $w_k^r = g^{rk} w_{kh}$.

On the other hand, from the second equation of (4.1), we get

$$(4.4) \quad L_X \nabla_j w_{ik} = -(L_X \{^h_{ji}\}) w_{hk} - (L_X \{^h_{ji}\}) w_{ih}.$$

Since the left side of (4.4) does not involve second derivatives of X , (4.4) is written by

$$(4.5) \quad w_{hk} \nabla_j \nabla_i X^h + w_{ih} \nabla_j \nabla_k X^h + \dots = 0.$$

Transvecting g^{ik} to the equation (4.5) we obtain

$$(4.6) \quad w_h^i \nabla_j \nabla_i X^h + \dots = 0.$$

Transvecting (4.5) with g^{ji} and making use of (4.6), we have

$$(4.7) \quad w_h^k g^{ji} \nabla_j \nabla_i X^h + \dots = 0.$$

Substituting (4.7) into the equation (4.3), we have

$$(4.8) \quad g^{ji} \nabla_j \nabla_i X^r + \dots = 0.$$

Thus we have a system of elliptic differential equation satisfied by all infinitesimal automorphisms X in $A(F, w)$. By Theorem of Bochner, $A(F, w)$ is finite-dimensional. This completes the proof of Theorem B.

5. Proof of Theorem C

By definition of an almost product structure, we are given a tensor field f such that $f^2=1$. This structure is a case of pseudo- f -structure of rank n . If there exists a positive definite Riemannian metric g such that

$$(5.1) \quad g(fX, fY) = g(X, Y)$$

for any vector fields X and Y on M , then the manifold M is called an almost product metric manifold.

We now define an automorphism on an almost product metric manifold M . A diffeomorphism h of M onto itself is called an automorphism of M if h leaves a covariant tensor field F invariant, that is

$$(5.2) \quad h^*F = F,$$

where the covariant vector field F is defined by

$$F(X, Y) = g(fX, Y)$$

for any vector fields X and Y on M . We denote the set of all automorphisms of M by $G(F)$. This group $G(F)$ is a case of the automorphism group $G(F, w)$ on a pseudo- f -manifold of rank n . Since the rank of f is n , we get $t=0$ and $w=0$. Therefore, by Theorem B, we can prove Theorem C.

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